

Statistics for low-lying zeros of symmetric power L -functions in the level aspect

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Communicated by Peter Sarnak

Abstract. We study one-level and two-level densities for low-lying zeros of symmetric power L -functions in the level aspect. This allows us to completely determine the symmetry types of some families of symmetric power L -functions with prescribed sign of functional equation. We also compute the moments of one-level density and exhibit mock-Gaussian behavior discovered by Hughes & Rudnick.

Keywords. L -functions, automorphic forms, symmetric powers, random matrix theory, statistics, zeroes.

2010 Mathematics Subject Classification. 11F66, 11F41, 11M50, 11M41.

1	Introduction and statement of the results	969
2	Automorphic and probabilistic background	980
3	Main technical ingredients of this work	988
4	Linear statistics for low-lying zeros	996
5	Quadratic statistics for low-lying zeros	1003
6	First asymptotic moments of the one-level density	1010
A	Analytic and arithmetic toolbox	1023
	Bibliography	1026

1 Introduction and statement of the results

1.1 Description of the families of L -functions studied

The purpose of this paper is to compute various statistics associated to low-lying zeros of several families of symmetric power L -functions in the level aspect. First of all, we give a short description of these families. To any primitive holomorphic

The first author is supported by the ANR project “Aspects Arithmétiques des Matrices Aléatoires et du Chaos Quantique”. The second author is supported by the ANR project “Modunombres”.

cuspidal form f of prime level q , trivial nebentypus and even weight¹ $\kappa \geq 2$ (see Section 2.1 for the automorphic background), say $f \in H_\kappa^*(q)$, one can associate its r -th symmetric power L -function denoted by $L(\text{Sym}^r f, s)$ for any integer $r \geq 1$. It is given by an explicit absolutely convergent Euler product of degree $r + 1$ on $\text{Re } s > 1$ (see Section 2.1.4). The completed L -function is defined by

$$\Lambda(\text{Sym}^r f, s) := (q^r)^{s/2} L_\infty(\text{Sym}^r f, s) L(\text{Sym}^r f, s)$$

where $L_\infty(\text{Sym}^r f, s)$ is a product of $r + 1$ explicit $\Gamma_{\mathbb{R}}$ -factors (see Section 2.1.4) and q^r is the arithmetic conductor. We will need some control on the analytic behavior of this function. Unfortunately, such information is not currently known in all generality. We sum up our main assumption in the following statement.

Hypothesis Nice(r, f). *The function $\Lambda(\text{Sym}^r f, s)$ is a completed L -function in the sense that it satisfies the following nice analytic properties:*

- *it can be extended to an holomorphic function of order 1 on \mathbb{C} ,*
- *it satisfies a functional equation of the shape*

$$\Lambda(\text{Sym}^r f, s) = \varepsilon(\text{Sym}^r f) \Lambda(\text{Sym}^r f, 1 - s)$$

where the sign $\varepsilon(\text{Sym}^r f) = \pm 1$ of the functional equation is given by

$$\varepsilon(\text{Sym}^r f) := \begin{cases} +1, & \text{if } r \text{ is even,} \\ \varepsilon_f(q) \times \varepsilon(\kappa, r), & \text{otherwise,} \end{cases} \tag{1.1}$$

with

$$\varepsilon(\kappa, r) := i^{\binom{r+1}{2}(\kappa-1) + \frac{r+1}{2}} = \begin{cases} i^\kappa, & \text{if } r \equiv 1 \pmod{8}, \\ -1, & \text{if } r \equiv 3 \pmod{8}, \\ -i^\kappa, & \text{if } r \equiv 5 \pmod{8}, \\ +1, & \text{if } r \equiv 7 \pmod{8}, \end{cases}$$

and $\varepsilon_f(q) = \pm 1$ is defined in (2.15) and only depends on f and q .

Remark 1.1. Hypothesis Nice(r, f) is known for $r = 1$ (E. Hecke [11, 12, 13]), $r = 2$ thanks to the work of S. Gelbart and H. Jacquet [9] and $r = 3, 4$ from the works of H. Kim and F. Shahidi [24, 23, 22]. In addition, Rankin–Selberg theory enabled H. Kim and F. Shahidi to prove the functional equation and the mero-

¹ In this paper, the weight κ is a *fixed* even integer and the level q goes to infinity among the prime numbers.

morphic continuation to \mathbb{C} for $k = 5, \dots, 9$ and the holomorphy in $\operatorname{Re} s \geq 1$ for $k = 5, \dots, 8$. Note that the holomorphy at $s = 1$ is expected since a holomorphic cusp form of prime level and trivial nebentypus is *not* of CM type. Very recently, a proof of the non-vanishing and the holomorphy of $L(\operatorname{Sym}^r f, s)$ on $\operatorname{Re} s \geq 1$ for any primitive holomorphic cusp form f of prime level q , trivial nebentypus and even weight f and any $r \geq 1$ has been announced by Barnet-Lamb, Geraghty, Harris & Taylor [2, Theorem B]².

We aim at studying the low-lying zeros for the family of L -functions given by

$$\mathcal{F}_r := \bigcup_{q \text{ prime}} \{L(\operatorname{Sym}^r f, s) : f \in H_\kappa^*(q)\}$$

for any integer $r \geq 1$. Note that when r is even, the sign of the functional equation of any $L(\operatorname{Sym}^r f, s)$ is constant of value $+1$ but when r is odd, this is definitely not the case. As a consequence, it is very natural to understand the low-lying zeros for the subfamilies given by

$$\mathcal{F}_r^\varepsilon := \bigcup_{q \text{ prime}} \{L(\operatorname{Sym}^r f, s) : f \in H_\kappa^*(q), \varepsilon(\operatorname{Sym}^r f) = \varepsilon\}$$

for any odd integer $r \geq 1$ and for $\varepsilon = \pm 1$.

1.2 Symmetry type of these families

One of the purposes of this work is to determine the symmetry type of the families \mathcal{F}_r and $\mathcal{F}_r^\varepsilon$ for $\varepsilon = \pm 1$ and for any integer $r \geq 1$ (see Section 4.1 for the background on symmetry types). The following theorem is a quick summary of the symmetry types obtained.

Theorem A. *Let $r \geq 1$ be any integer and $\varepsilon = \pm 1$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any primitive holomorphic cusp form of level q and even weight $\kappa \geq 2$. The symmetry group $G(\mathcal{F}_r)$ of \mathcal{F}_r is given by*

$$G(\mathcal{F}_r) = \begin{cases} \operatorname{Sp}, & \text{if } r \text{ is even,} \\ \operatorname{O}, & \text{otherwise.} \end{cases}$$

If r is odd, then the symmetry group $G(\mathcal{F}_r^\varepsilon)$ of $\mathcal{F}_r^\varepsilon$ is given by

$$G(\mathcal{F}_r^\varepsilon) = \begin{cases} \operatorname{SO}(\text{even}), & \text{if } \varepsilon = +1, \\ \operatorname{SO}(\text{odd}), & \text{otherwise.} \end{cases}$$

² We refer the reader to Hypothesis $\operatorname{Sym}^r(f)$ in Section 2 for additional comments.

Remark 1.2. In this paper, when we say (for shortness) that a family \mathcal{F} of L -functions has symmetry type $G \in \{O, SO(\text{even}), SO(\text{odd}), Sp, U\}$, we actually say that if \mathcal{F} has a symmetry type in $\{O, SO(\text{even}), SO(\text{odd}), Sp, U\}$, then it is G . In other words, our aim is to describe the symmetry type of a family, assuming it has one and that this one is among the classical compact groups.

Remark 1.3. Note that we do not assume any Generalised Riemann Hypothesis for the symmetric power L -functions.

In order to prove Theorem A, we compute either the (signed) asymptotic expectation of the one-level density or the (signed) asymptotic expectation of the two-level density (it has been noticed by S. J. Miller [28] that this is sufficient to distinguish the symmetry types). The results are given in the next two sections in which $\varepsilon = \pm 1$, ν will always be a positive real number, Φ , Φ_1 and Φ_2 will always stand for even Schwartz functions whose Fourier transforms $\widehat{\Phi}$, $\widehat{\Phi}_1$ and $\widehat{\Phi}_2$ are compactly supported in $[-\nu, +\nu]$ and f will always be a primitive holomorphic cusp form of prime level q and even weight $\kappa \geq 2$ for which Hypothesis Nice(r, f) holds. We refer to Section 2.2 for the probabilistic background.

1.2.1 (Signed) asymptotic expectation of the one-level density

The *one-level density* (relatively to Φ) of $\text{Sym}^r f$ is defined by

$$D_{1,q}[\Phi; r](f) := \sum_{\rho, \Lambda(\text{Sym}^r f, \rho)=0} \Phi \left(\frac{\log(q^r)}{2i\pi} \left(\text{Re } \rho - \frac{1}{2} + i \text{Im } \rho \right) \right)$$

where the sum is over the non-trivial zeros ρ of $L(\text{Sym}^r f, s)$ with multiplicities. The *asymptotic expectation* of the one-level density is by definition

$$\lim_{\substack{q \text{ prime} \\ q \rightarrow +\infty}} \sum_{f \in H_\kappa^*(q)} \omega_q(f) D_{1,q}[\Phi; r](f)$$

where $\omega_q(f)$ is the harmonic weight defined in (2.7) and similarly the *signed asymptotic expectation* of the one-level density is by definition

$$\lim_{\substack{q \text{ prime} \\ q \rightarrow +\infty}} 2 \sum_{\substack{f \in H_\kappa^*(q) \\ \varepsilon(\text{Sym}^r f)=\varepsilon}} \omega_q(f) D_{1,q}[\Phi; r](f)$$

when r is odd.

Theorem B. *Let $r \geq 1$ be any integer and $\varepsilon = \pm 1$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any primitive holomorphic cusp form of level q and even weight $\kappa \geq 2$ and also that θ is admissible (see Hypothe-*

sis $H_2(\theta)$, page 989). Let

$$\nu_{1,\max}(r, \kappa, \theta) := \left(1 - \frac{1}{2(\kappa - 2\theta)}\right) \frac{2}{r^2}.$$

If $\nu < \nu_{1,\max}(r, \kappa, \theta)$, then the asymptotic expectation of the one-level density is

$$\widehat{\Phi}(0) + \frac{(-1)^{r+1}}{2} \Phi(0).$$

Let

$$\nu_{1,\max}^\varepsilon(r, \kappa, \theta) := \inf \left(\nu_{1,\max}(r, \kappa, \theta), \frac{3}{r(r+2)} \right).$$

If r is odd and $\nu < \nu_{1,\max}^\varepsilon(r, \kappa, \theta)$, then the signed asymptotic expectation of the one-level density is

$$\widehat{\Phi}(0) + \frac{(-1)^{r+1}}{2} \Phi(0).$$

Remark 1.4. The first part of Theorem B reveals that the symmetry type of \mathcal{F}_r is

$$G(\mathcal{F}_r) = \begin{cases} \text{Sp}, & \text{if } r \text{ is even,} \\ \text{O}, & \text{if } r = 1, \\ \text{SO(even) or O or SO(odd),} & \text{if } r \geq 3 \text{ is odd.} \end{cases}$$

We cannot decide between the three orthogonal groups when $r \geq 3$ is odd since in this case $\nu_{1,\max}(r, \kappa, \theta) < 1$ but the computation of the two-level densities will enable us to decide. Note also that we go beyond the support $[-1, 1]$ when $r = 1$ as Iwaniec, Luo & Sarnak [20, Theorem 1.1] but without doing any subtle arithmetic analysis of Kloosterman sums. Also, A. Güloğlu established in [10, Theorem 1.2] some density result for the same family of L -functions but when the weight κ goes to infinity and the level q is fixed. It turns out that we recover the same constraint on ν when r is even but we get a better result when r is odd. This can be explained by the fact that the analytic conductor of any $L(\text{Sym}^r f, s)$ with f in $H_\kappa^*(q)$, which is of size

$$q^r \times \begin{cases} \kappa^r, & \text{if } r \text{ is even,} \\ \kappa^{r+1}, & \text{otherwise,} \end{cases}$$

is slightly larger in his case than in ours when r is odd.

Remark 1.5. The second part of Theorem B reveals that if r is odd and $\varepsilon = \pm 1$, then the symmetry type of $\mathcal{F}_r^\varepsilon$ is

$$G(\mathcal{F}_r^\varepsilon) = \text{SO(even) or O or SO(odd)}.$$

Here ν is always strictly smaller than one and we are not able to recover the result of [20, Theorem 1.1] without doing some arithmetic on Kloosterman sums.

1.2.2 Sketch of the proof

We give here a sketch of the proof of the first part of Theorem B namely we briefly explain how to determine the asymptotic expectation of the one-level density assuming that Hypothesis Nice(r, f) holds for any prime number q and any primitive holomorphic cusp form of level q and even weight $\kappa \geq 2$ and also that θ is admissible. The first step consists in transforming the sum over the zeros of $\Lambda(\text{Sym}^r f, s)$, which occurs in $D_{1,q}[\Phi; r](f)$ into a sum over primes. This is done *via* some Riemann’s explicit formula for symmetric power L -functions (see Proposition 3.8), which leads to

$$D_{1,q}[\Phi; r](f) = \widehat{\Phi}(0) + \frac{(-1)^{r+1}}{2} \Phi(0) + P_q^1[\Phi; r](f) + \sum_{m=0}^{r-1} (-1)^m P_q^2[\Phi; r, m](f) + o(1)$$

where

$$P_q^1[\Phi; r](f) := -\frac{2}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \lambda_f(p^r) \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right). \tag{1.2}$$

The terms $P_q^2[\Phi; r, m](f)$ are also sums over primes, which look like $P_q^1[\Phi; r](f)$. They can be forgotten in first approximation. Indeed they can be thought as sums over squares of primes hence are easier to deal with. The second step consists in averaging over all the f in $H_\kappa^*(q)$. While doing this, the asymptotic expectation of the one-level density

$$\widehat{\Phi}(0) + \frac{(-1)^{r+1}}{2} \Phi(0)$$

naturally appears and we need to show that

$$-\frac{2}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \left(\sum_{f \in H_\kappa^*(q)} \omega_q(f) \lambda_f(p^r) \right) \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right)$$

is a remainder term provided that the support ν of Φ is small enough. We apply some suitable trace formula (see Proposition 2.2) in order to express the previous average of Hecke eigenvalues. We cannot directly apply Petersson’s trace formula since there may be some old forms of level q especially when the weight κ is large. Nevertheless, these old forms are automatically of level 1 since q is prime and their

contribution remains negligible. So, we have to bound

$$-\frac{4\pi i^\kappa}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \sum_{\substack{c \geq 1 \\ q \mid c}} \frac{S(1, p^r; c)}{c} J_{\kappa-1} \left(\frac{4\pi \sqrt{p^r}}{c} \right) \frac{\log p}{\sqrt{p}} \widehat{\Phi} \left(\frac{\log p}{\log(q^r)} \right)$$

where $S(1, p^r; c)$ is a Kloosterman sum and which can be written as

$$-\frac{4\pi i^\kappa}{\log(q^r)} \sum_{\substack{c \geq 1 \\ q \mid c}} \sum_{m \geq 1} a_m \frac{S(1, m; c)}{c} g(m; c)$$

where

$$a_m := \mathbb{1}_{[1, q^{r^2\nu}]}(m) \frac{\log m}{rm^{1/(2r)}} \times \begin{cases} 1, & \text{if } m = p^r \text{ for some prime } p \neq q, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$g(m; c) := J_{\kappa-1} \left(\frac{4\pi \sqrt{m}}{c} \right) \widehat{\Phi} \left(\frac{\log m}{r \log(q^r)} \right).$$

We apply the large sieve inequality for Kloosterman sums given in Proposition 3.4. It implies that if $\nu \leq 2/r^2$, then our quantity is bounded by

$$\ll_\varepsilon q^{(\frac{\kappa-1}{2}-\theta)(r^2\nu-2)+\varepsilon} + q^{(\frac{\kappa}{2}-\theta)r^2\nu-(\kappa-\frac{1}{2}-2\theta)+\varepsilon}.$$

This is an admissible error term if $\nu < \nu_{1, \max}(r, \kappa, \theta)$. We focus on the fact that we did not do any arithmetic analysis of Kloosterman sums to get this result. Of course, the power of spectral theory of automorphic forms is hidden in the large sieve inequalities for Kloosterman sums.

1.2.3 (Signed) asymptotic expectation of the two-level density

The *two-level density* of $\text{Sym}^r f$ (relatively to Φ_1 and Φ_2) is defined by

$$D_{2,q}[\Phi_1, \Phi_2; r](f) := \sum_{\substack{(j_1, j_2) \in \mathcal{E}(f, r)^2 \\ j_1 \neq \pm j_2}} \Phi_1 \left(\widehat{\rho}_{f,r}^{(j_1)} \right) \Phi_2 \left(\widehat{\rho}_{f,r}^{(j_2)} \right).$$

For more precision on the numbering of the zeros, we refer to Section 3.2. The *asymptotic expectation* of the two-level density is by definition

$$\lim_{\substack{q \text{ prime} \\ q \rightarrow +\infty}} \sum_{f \in H_\kappa^*(q)} \omega_q(f) D_{2,q}[\Phi_1, \Phi_2; r](f)$$

and similarly the *signed asymptotic expectation* of the two-level density is by definition

$$\lim_{\substack{q \text{ prime} \\ q \rightarrow +\infty}} 2 \sum_{\substack{f \in H_{\kappa}^*(q) \\ \varepsilon(\text{Sym}^r f) = \varepsilon}} \omega_q(f) D_{2,q}[\Phi_1, \Phi_2; r](f)$$

when r is odd and $\varepsilon = \pm 1$.

Theorem C. *Let $r \geq 1$ be any integer and $\varepsilon = \pm 1$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any primitive holomorphic cusp form of level q and even weight $\kappa \geq 2$. If $\nu < 1/r^2$, then the asymptotic expectation of the two-level density is*

$$\begin{aligned} & \left[\widehat{\Phi}_1(0) + \frac{(-1)^{r+1}}{2} \Phi_1(0) \right] \left[\widehat{\Phi}_2(0) + \frac{(-1)^{r+1}}{2} \Phi_2(0) \right] \\ & + 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) \, du - 2 \widehat{\Phi}_1 \widehat{\Phi}_2(0) \\ & + \left((-1)^r + \frac{\mathbb{1}_{2\mathbb{N}+1}(r)}{2} \right) \Phi_1(0) \Phi_2(0). \end{aligned}$$

If r is odd and $\nu < 1/(2r(r+2))$, then the signed asymptotic expectation of the two-level density is

$$\begin{aligned} & \left[\widehat{\Phi}_1(0) + \frac{1}{2} \Phi_1(0) \right] \left[\widehat{\Phi}_2(0) + \frac{1}{2} \Phi_2(0) \right] \\ & + 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) \, du - 2 \widehat{\Phi}_1 \widehat{\Phi}_2(0) - \Phi_1(0) \Phi_2(0) \\ & + \mathbb{1}_{\{-1\}}(\varepsilon) \Phi_1(0) \Phi_2(0). \end{aligned}$$

Remark 1.6. We have just seen that the computation of the one-level density already reveals that the symmetry type of \mathcal{F}_r is Sp when r is even. The asymptotic expectation of the two-level density also coincides with the one of Sp (see [21, Theorem A.D.2.2] or [28, Theorem 3.3]). When $r \geq 3$ is odd, the first part of Theorem C together with a result of Katz & Sarnak (see [21, Theorem A.D.2.2] or [28, Theorem 3.2]) imply that the symmetry type of \mathcal{F}_r is O .

Remark 1.7. The second part of Theorem C and a result of Katz & Sarnak (see [21, Theorem A.D.2.2] or [28, Theorem 3.2]) imply that the symmetry type of $\mathcal{F}_r^\varepsilon$ is as in Theorem A for any odd integer $r \geq 1$ and $\varepsilon = \pm 1$.

In order to prove Theorem C, we need to determine the *asymptotic variance* of the one-level density, which is defined by

$$\lim_{\substack{q \text{ prime} \\ q \rightarrow +\infty}} \sum_{f \in H_{\kappa}^*(q)} \omega_q(f) \left(D_{1,q}[\Phi; r](f) - \sum_{g \in H_{\kappa}^*(q)} \omega_q(g) D_{1,q}[\Phi; r](g) \right)^2$$

and the *signed asymptotic variance* of the one-level density, which is similarly defined by

$$\lim_{\substack{q \text{ prime} \\ q \rightarrow +\infty}} 2 \sum_{\substack{f \in H_{\kappa}^*(q) \\ \varepsilon(\text{Sym}^r f) = \varepsilon}} \omega_q(f) \left(D_{1,q}[\Phi; r](f) - 2 \sum_{\substack{g \in H_{\kappa}^*(q) \\ \varepsilon(\text{Sym}^r g) = \varepsilon}} \omega_q(g) D_{1,q}[\Phi; r](g) \right)^2$$

when r is odd and $\varepsilon = \pm 1$.

Theorem D. *Let $r \geq 1$ be any integer and $\varepsilon = \pm 1$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any primitive holomorphic cusp form of level q and even weight $\kappa \geq 2$. If $\nu < 1/r^2$, then the asymptotic variance of the one-level density is*

$$2 \int_{\mathbb{R}} |u| \widehat{\Phi}^2(u) \, du.$$

If r is odd and $\nu < 1/(2r(r + 2))$, then the signed asymptotic variance of the one-level density is

$$2 \int_{\mathbb{R}} |u| \widehat{\Phi}^2(u) \, du.$$

1.3 Asymptotic moments of the one-level density

Last but not least, we compute the *asymptotic m -th moment* of the one-level density, which is defined by

$$\lim_{\substack{q \text{ prime} \\ q \rightarrow +\infty}} \sum_{f \in H_{\kappa}^*(q)} \omega_q(f) \left(D_{1,q}[\Phi; r](f) - \sum_{g \in H_{\kappa}^*(q)} \omega_q(g) D_{1,q}[\Phi; r](g) \right)^m$$

for any integer $m \geq 1$.

Theorem E. *Let $r \geq 1$ be any integer and $\varepsilon = \pm 1$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any primitive holomorphic cusp form of level q and even weight $\kappa \geq 2$. If $m\nu < 4/(r(r + 2))$, then the asymptotic*

m-th moment of the one-level density is

$$\begin{cases} 0, & \text{if } m \text{ is odd,} \\ \left(2 \int_{\mathbb{R}} |u| \widehat{\Phi}^2(u) \, du\right)^{m/2} \times \frac{m!}{2^{m/2} \left(\frac{m}{2}\right)!}, & \text{otherwise.} \end{cases}$$

Remark 1.8. This result seems to exhibit Gaussian behavior. If we were able to increase the admissible value of mv , mock-Gaussian would probably appear (see [14, 16, 17, 15] for instance).

Remark 1.9. In [15], Hughes & Miller evaluate the moments of the one-level density in the case $r = 1$ on a quite large support that allows to exhibit mock-Gaussian behavior. They need to consider non-diagonal terms, which makes the combinatorial analysis difficult.

Remark 1.10. We compute the first asymptotic moments of the one-level density. These computations allow to compute the asymptotic expectation of the first level-densities [14, §1.2]. We will use the specific case of the asymptotic expectation of the two-level density and the asymptotic variance in Section 5.1.

Let us sketch the proof of Theorem E by explaining the origin of the main term. We have to evaluate

$$\sum_{0 \leq \ell \leq m} \binom{m}{\ell} \binom{\ell}{\alpha} \mathbb{E}_q^h \left(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell \right) \tag{1.3}$$

where $P_q^1[\Phi; r]$ has been defined in (1.2),

$$P_q^2[\Phi; r](f) = -\frac{2}{\log(q^r)} \sum_{j=1}^r (-1)^{r-j} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \lambda_f(p^{2j}) \frac{\log p}{p} \widehat{\Phi} \left(\frac{2 \log p}{\log(q^r)} \right).$$

The main term comes from the contribution $\ell = 0$ in the sum (1.3). Using a combinatorial lemma, we rewrite this main contribution as

$$\frac{(-2)^m}{\log^m(q^r)} \sum_{s=1}^m \sum_{\sigma \in P(m,s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \mathbb{E}_q^h \left(\prod_{u=1}^s \lambda_f(\widehat{p}_{i_u}^r)^{\omega_u^{(\sigma)}} \right)$$

where $P(m, s)$ is the set of surjective functions

$$\sigma: \{1, \dots, \alpha\} \twoheadrightarrow \{1, \dots, s\}$$

such that for any $j \in \{1, \dots, s\}$, either $\sigma(j) = 1$ or there exists $k < j$ such that $\sigma(j) = \sigma(k) + 1$ and

$$\varpi_j^{(\sigma)} := \#\sigma^{-1}(\{j\})$$

for any $j \in \{1, \dots, s\}$. We write $(\widehat{p}_i)_{i \geq 1}$ stands for the increasing sequence of prime numbers different from q . Linearising each $\lambda_f(\widehat{p}_{i_u}^r)^{\varpi_u^{(\sigma)}}$ in terms of $\lambda_f(\widehat{p}_{i_u}^{j_u})$ with j_u runs over integers in $[0, r \varpi_u^{(\sigma)}]$ and using a trace formula to prove that the only $\sigma \in P(m, s)$ leading to a principal contribution satisfy $\varpi_j^{(\sigma)} = 2$ for any $j \in \{1, \dots, s\}$, we have to estimate

$$\frac{(-2)^m}{\log^m(q^r)} \sum_{s=1}^m \sum_{\substack{\sigma \in P(m,s) \\ \forall j \in \{1, \dots, s\}, \varpi_j^{(\sigma)} = 2}} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \frac{\log^2(\widehat{p}_{i_u})}{\widehat{p}_{i_u}} \widehat{\Phi}^2\left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)}\right). \tag{1.4}$$

This sum vanishes if m is odd since

$$\sum_{j=1}^s \varpi_j^{(\sigma)} = m$$

and it remains to prove the formula for m even. In this case, and since we already computed the moment for $m = 2$, we deduce from (1.4) that the main contribution is

$$\mathbb{E}_q^h(P_q^1[\Phi; r]^2) \times \#\left\{\sigma \in P(m, m/2) : \varpi_j^{(\sigma)} = 2 (\forall j)\right\}$$

and we conclude by computing

$$\#\left(\left\{\sigma \in P(m, m/2) : \varpi_j^{(\sigma)} = 2 (\forall j)\right\}\right) = \frac{m!}{2^{m/2} \left(\frac{m}{2}\right)!}.$$

Proving that the other terms lead to error terms is done by implementing similar ideas, but requires – especially for the double products (namely terms implying both P_q^1 and P_q^2) – much more combinatorial technicalities.

1.4 Organisation of the paper

Section 2 contains the automorphic and probabilistic background, which is needed to be able to read this paper. In particular, we give here the accurate definition of symmetric power L -functions and the properties of Chebyshev polynomials useful in Section 6. In Section 3 we describe the main technical ingredients of this work namely large sieve inequalities for Kloosterman sums and Riemann’s ex-

PLICIT formula for symmetric power L -functions. In Section 4, some standard facts about symmetry groups are given and the computation of the (signed) asymptotic expectation of the one-level density is done. The computations of the (signed) asymptotic expectation, covariance and variance of the two-level density are done in Section 5 whereas the computation of the asymptotic moments of the one-level density is provided in Section 6. Some well-known facts about Kloosterman sums are recalled in Appendix A.

Notation. We write \mathcal{P} for the set of prime numbers and the main parameter in this paper is a prime number q , whose name is the level, which goes to infinity among \mathcal{P} . Thus, if f and g are some \mathbb{C} -valued functions of the real variable, then the notations $f(q) \ll_A g(q)$ or $f(q) = O_A(g(q))$ mean that $|f(q)|$ is smaller than a “constant”, which only depends on A times $g(q)$, at least for q a large enough prime number, and similarly, $f(q) = o(1)$ means that $f(q) \rightarrow 0$ as q goes to infinity among the prime numbers. We will denote by ε an absolute positive constant whose definition may vary from one line to the next one. The characteristic function of a set S will be denoted $\mathbb{1}_S$. Finally, $\tau(\ell)$ stands for the number of divisors of ℓ and $\ell \mid q^\infty$ means that the integer ℓ has the same prime divisors than q .

2 Automorphic and probabilistic background

2.1 Automorphic background

2.1.1 Overview of holomorphic cusp forms

In this section, we recall general facts about holomorphic cusp forms. A reference is [18].

Generalities. We write $\Gamma_0(q)$ for the congruence subgroup of level q , which acts on the upper-half plane \mathcal{H} . A holomorphic function $f: \mathcal{H} \mapsto \mathbb{C}$, which satisfies

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q), \forall z \in \mathcal{H}, \quad f\left(\frac{az + b}{cz + d}\right) = (cz + d)^\kappa f(z)$$

and vanishes at the cusps of $\Gamma_0(q)$ is a *holomorphic cusp form* of level q , even weight $\kappa \geq 2$. We denote by $S_\kappa(q)$ this space of holomorphic cusp forms, which is equipped with the Petersson inner product

$$\langle f_1, f_2 \rangle_q := \int_{\Gamma_0(q) \backslash \mathcal{H}} y^\kappa f_1(z) \overline{f_2(z)} \frac{dx dy}{y^2}.$$

The Fourier expansion at the cusp ∞ of any such holomorphic cusp form f is given by

$$\forall z \in \mathcal{H}, \quad f(z) = \sum_{n \geq 1} \psi_f(n) n^{(\kappa-1)/2} e(nz)$$

where $e(z) := \exp(2i\pi z)$ for any complex number z . The Hecke operators act on $S_\kappa(q)$ by

$$T_\ell(f)(z) := \frac{1}{\sqrt{\ell}} \sum_{\substack{ad=\ell \\ (a,q)=1}} \sum_{0 \leq b < d} f\left(\frac{az+b}{d}\right)$$

for any $z \in \mathcal{H}$. If f is an eigenvector of T_ℓ , we write $\lambda_f(\ell)$ for the corresponding eigenvalue. One can prove that T_ℓ is hermitian if $\ell \geq 1$ is any integer coprime with q , and

$$T_{\ell_1} \circ T_{\ell_2} = \sum_{\substack{d | (\ell_1, \ell_2) \\ (d,q)=1}} T_{\ell_1 \ell_2 / d^2} \tag{2.1}$$

for any integers $\ell_1, \ell_2 \geq 1$. By Atkin–Lehner theory [1], we get a splitting of $S_\kappa(q)$ into $S_\kappa^o(q) \oplus^{\perp(\cdot, \cdot)_q} S_\kappa^n(q)$ where

$$\begin{aligned} S_\kappa^o(q) &:= \text{Vect}_{\mathbb{C}} \{f(qz) : f \in S_\kappa(1)\} \oplus S_\kappa(1), \\ S_\kappa^n(q) &:= (S_\kappa^o(q))^{\perp(\cdot, \cdot)_q} \end{aligned}$$

where “o” stands for “old” and “n” for “new”. Note that $S_\kappa^o(q) = \{0\}$ if $\kappa < 12$ or $\kappa = 14$. These two spaces are T_ℓ -invariant for any integer $\ell \geq 1$ coprime with q . A primitive cusp form $f \in S_\kappa^n(q)$ is an eigenfunction of any operator T_ℓ for any integer $\ell \geq 1$ coprime with q , which is new and arithmetically normalised, namely $\psi_f(1) = 1$. Such an element f is automatically an eigenfunction of the other Hecke operators and satisfies $\psi_f(\ell) = \lambda_f(\ell)$ for any integer $\ell \geq 1$. Moreover, if p is a prime number, define $\alpha_f(p), \beta_f(p)$ as the complex roots of the quadratic equation

$$X^2 - \lambda_f(p)X + \varepsilon_q(p) = 0 \tag{2.2}$$

where ε_q denotes the trivial Dirichlet character of modulus q . Then it follows from the work of Eichler, Shimura, Igusa and Deligne that

$$|\alpha_f(p)| = |\beta_f(p)| = 1$$

for any prime number p and so

$$\forall \ell \geq 1, \quad |\lambda_f(\ell)| \leq \tau(\ell). \tag{2.3}$$

The set of primitive cusp forms is denoted by $H_\kappa^*(q)$. It is an orthogonal basis of $S_\kappa^n(q)$. Let f be a holomorphic cusp form with Hecke eigenvalues $(\lambda_f(\ell))_{(\ell,q)=1}$. The composition property (2.1) entails that for any integer $\ell_1 \geq 1$ and for any integer $\ell_2 \geq 1$ coprime with q the following multiplicative relations hold:

$$\psi_f(\ell_1)\lambda_f(\ell_2) = \sum_{\substack{d | (\ell_1, \ell_2) \\ (d, q) = 1}} \psi_f(\ell_1\ell_2/d^2), \tag{2.4}$$

$$\psi_f(\ell_1\ell_2) = \sum_{\substack{d | (\ell_1, \ell_2) \\ (d, q) = 1}} \mu(d)\psi_f(\ell_1/d)\lambda_f(\ell_2/d) \tag{2.5}$$

and these relations hold for any integers $\ell_1, \ell_2 \geq 1$ if f is primitive. The adjointness relation is

$$\lambda_f(\ell) = \overline{\lambda_f(\ell)}, \quad \psi_f(\ell) = \overline{\psi_f(\ell)} \tag{2.6}$$

for any integer $\ell \geq 1$ coprime with q and this remains true for any integer $\ell \geq 1$ if f is primitive.

Trace formulas. We need two definitions. The harmonic weight associated to any f in $S_\kappa(q)$ is defined by

$$\omega_q(f) := \frac{\Gamma(\kappa - 1)}{(4\pi)^\kappa \langle f, f \rangle_q}. \tag{2.7}$$

For any natural integer m and n , the Δ_q -symbol is given by

$$\Delta_q(m, n) := \delta_{m,n} + 2\pi i^\kappa \sum_{\substack{c \geq 1 \\ q | c}} \frac{S(m, n; c)}{c} J_{\kappa-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{2.8}$$

where $S(m, n; c)$ is a Kloosterman sum defined in Appendix A.3 and $J_{\kappa-1}$ is a Bessel function of first kind defined in Appendix A.2. The following proposition is the *Petersson trace formula*.

Proposition 2.1. *If $H_\kappa(q)$ is any orthogonal basis of $S_\kappa(q)$, then*

$$\sum_{f \in H_\kappa(q)} \omega_q(f)\psi_f(m)\psi_f(n) = \Delta_q(m, n) \tag{2.9}$$

for any integers m and n .

H. Iwaniec, W. Luo and P. Sarnak proved in [20] a useful variation of the Petersson trace formula, which is an average over only primitive cusp forms. This

is more convenient when there are some old forms, which is the case for instance when the weight κ is large. Let ν be the arithmetic function defined by

$$\nu(n) := n \prod_{p|n} (1 + 1/p)$$

for any integer $n \geq 1$.

Proposition 2.2 (H. Iwaniec, W. Luo & P. Sarnak (2001)). *If $(n, q^2) \mid q$ and $q \nmid m$, then*

$$\sum_{f \in H_{\kappa}^*(q)} \omega_q(f) \lambda_f(m) \lambda_f(n) = \Delta_q(m, n) - \frac{1}{q\nu((n, q))} \sum_{\ell \mid q^\infty} \frac{1}{\ell} \Delta_1(m\ell^2, n). \tag{2.10}$$

Remark 2.3. The first term in (2.10) is exactly the term which appears in (2.9) whereas the second term in (2.10) will be usually very small as an old form comes from a form of level 1! Thus, everything will work in practice as if there were no old forms in $S_{\kappa}(q)$.

2.1.2 Chebyshev polynomials and Hecke eigenvalues

Let $p \neq q$ be a prime number and $f \in H_{\kappa}^*(q)$. The multiplicativity relation (2.4) leads to

$$\sum_{r \geq 0} \lambda_f(p^r) t^r = \frac{1}{1 - \lambda_f(p)t + t^2}.$$

It follows that

$$\lambda_f(p^r) = X_r(\lambda_f(p)) \tag{2.11}$$

where the polynomials X_r are defined by their generating series

$$\sum_{r \geq 0} X_r(x) t^r = \frac{1}{1 - xt + t^2}.$$

They are also defined by

$$X_r(2 \cos \theta) = \frac{\sin((r + 1)\theta)}{\sin(\theta)}.$$

These polynomials are known as the Chebyshev polynomials of the second kind. Each X_r has degree r , is even if r is even and odd otherwise. The family $\{X_r\}_{r \geq 0}$ is a basis for $\mathbb{Q}[X]$, orthonormal with respect to the inner product

$$\langle P, Q \rangle_{\text{ST}} := \frac{1}{\pi} \int_{-2}^2 P(x) Q(x) \sqrt{1 - \frac{x^2}{4}} \, dx$$

where “ST” stands for Sato–Tate. In particular, for any integer $\varpi \geq 0$ we have

$$X_r^\varpi = \sum_{j=0}^{r\varpi} x(\varpi, r, j) X_j \tag{2.12}$$

with

$$x(\varpi, r, j) := \langle X_r^\varpi, X_j \rangle_{\text{ST}} = \frac{2}{\pi} \int_0^\pi \frac{\sin^\varpi((r+1)\theta) \sin((j+1)\theta)}{\sin^{\varpi-1}(\theta)} d\theta. \tag{2.13}$$

The following relations are useful in this paper:

$$x(\varpi, r, j) = \begin{cases} 1, & \text{if } j = 0 \text{ and } \varpi = 2, \\ 0, & \text{if } j \text{ is odd and } r \text{ is even,} \\ 0, & \text{if } j = 0, \varpi = 1 \text{ and } r \geq 1. \end{cases} \tag{2.14}$$

2.1.3 Overview of L -functions associated to primitive cusp forms

Let f be in $H_\kappa^*(q)$. We define

$$L(f, s) := \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_{p \in \mathcal{P}} \left(1 - \frac{\alpha_f(p)}{p^s} \right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s} \right)^{-1},$$

which is an absolutely convergent and non-vanishing Dirichlet series and Euler product on $\text{Re } s > 1$, and also

$$L_\infty(f, s) := \Gamma_{\mathbb{R}}(s + (\kappa - 1)/2) \Gamma_{\mathbb{R}}(s + (\kappa + 1)/2)$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2} \Gamma(s/2)$ as usual. The function

$$\Lambda(f, s) := q^{s/2} L_\infty(f, s) L(f, s)$$

is a *completed L -function* in the sense that it satisfies the following *nice* analytic properties:

- the function $\Lambda(f, s)$ can be extended to a holomorphic function of order 1 on \mathbb{C} ,
- the function $\Lambda(f, s)$ satisfies a functional equation of the shape

$$\Lambda(f, s) = i^\kappa \varepsilon_f(q) \Lambda(f, 1 - s)$$

where

$$\varepsilon_f(q) = -\sqrt{q} \lambda_f(q) = \pm 1. \tag{2.15}$$

2.1.4 Overview of symmetric power L -functions

Let f be in $H_\kappa^*(q)$. For any natural integer $r \geq 1$, the symmetric r -th power associated to f is given by the following Euler product of degree $r + 1$:

$$L(\text{Sym}^r f, s) := \prod_{p \in \mathcal{P}} L_p(\text{Sym}^r f, s)$$

where

$$L_p(\text{Sym}^r f, s) := \prod_{i=0}^r \left(1 - \frac{\alpha_f(p)^i \beta_f(p)^{r-i}}{p^s} \right)^{-1}$$

for any prime number p . Let us remark that the local factors of this Euler product may be written as

$$L_p(\text{Sym}^r f, s) = \prod_{i=0}^r \left(1 - \frac{\alpha_f(p)^{2i-r}}{p^s} \right)^{-1}$$

for any prime number $p \neq q$ and

$$L_q(\text{Sym}^r f, s) = \left(1 - \frac{\lambda_f(q)^r}{q^s} \right)^{-1} = \left(1 - \frac{\lambda_f(q^r)}{q^s} \right)^{-1},$$

since $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = \varepsilon_q(p)$ for any prime number p according to (2.2). On $\text{Re } s > 1$, this Euler product is absolutely convergent and non-vanishing. We also define a local factor at ∞ (see [5, (3.16) and (3.17)]), which is given by a product of $r + 1$ Gamma factors namely

$$L_\infty(\text{Sym}^r f, s) := \prod_{0 \leq a \leq (r-1)/2} \left(\Gamma_{\mathbb{R}}(s + (2a + 1)(\kappa - 1)/2) \right. \\ \left. \times \Gamma_{\mathbb{R}}(s + 1 + (2a + 1)(\kappa - 1)/2) \right)$$

if r is odd and

$$L_\infty(\text{Sym}^r f, s) := \Gamma_{\mathbb{R}}(s + \mu_{\kappa,r}) \prod_{1 \leq a \leq r/2} \Gamma_{\mathbb{R}}(s + a(\kappa - 1)) \Gamma_{\mathbb{R}}(s + 1 + a(\kappa - 1))$$

if r is even where

$$\mu_{\kappa,r} := \begin{cases} 1, & \text{if } r(\kappa - 1)/2 \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

All the local data appearing in these local factors are encapsulated in the following completed L -function:

$$\Lambda(\text{Sym}^r f, s) := (q^r)^{s/2} L_\infty(\text{Sym}^r f, s) L(\text{Sym}^r f, s).$$

Here, q^r is called the arithmetic conductor of $\Lambda(\text{Sym}^r f, s)$ and somehow measures the size of this function. We will need some control on the analytic behavior of this function. Unfortunately, such information is not currently known in all generality. Our main assumption is given in Hypothesis Nice(r, f), page 970. Indeed, much more is expected to hold (confer the detailed discussion in [5]) namely the following assumption, which lies in the spirit of Langlands' program, is strongly believed to be true.

Hypothesis $\text{Sym}^r(f)$. *There exists an automorphic cuspidal self-dual representation, denoted by $\text{Sym}^r \pi_f = \bigotimes'_{p \in \mathcal{P} \cup \{\infty\}} \text{Sym}^r \pi_{f,p}$, of $\text{GL}_{r+1}(\mathbb{A}_{\mathbb{Q}})$ whose local factors $L(\text{Sym}^r \pi_{f,p}, s)$ agree with the local factors $L_p(\text{Sym}^r f, s)$ for any $p \in \mathcal{P} \cup \{\infty\}$.*

Note that the local factors and the arithmetic conductor in the definition of $\Lambda(\text{Sym}^r f, s)$ and also the sign of its functional equation, which all appear without any explanations so far, come from the explicit computations, which have been done *via* the local Langlands correspondence by J. Cogdell and P. Michel in [5].

Hypothesis Nice(r, f) is obviously a weak consequence of Hypothesis $\text{Sym}^r(f)$ and the fact that there is *no* CM forms of prime level and trivial nebentypus. For instance, the cuspidality condition in Hypothesis $\text{Sym}^r(f)$ entails the fact that $\Lambda(\text{Sym}^r f, s)$ is of order 1, which is crucial for us to state a suitable explicit formula. Nevertheless, one should keep in mind that it could happen that one knows the expected analytic properties of a general L -function whereas the modularity of this L -function remains an open question.

It turns out that Barnet-Lamb, Geraghty, Harris & Taylor [2] have recently announced a proof for the potential automorphy³ of $L(\text{Sym}^r f, s)$, which means that there exists such an automorphic cuspidal self-dual representation but over a Galois totally real extension of \mathbb{Q} .

Some additional comments are given in Remark 1.1, page 970.

2.2 Probabilistic background

The set $H_{\kappa}^*(q)$ can be seen as a probability space if

- the measurable sets are all its subsets,
- the *harmonic probability measure* is defined by

$$\mu_q^h(A) := \sum_{f \in A}^h 1 := \sum_{f \in A} \omega_q(f)$$

for any subset A of $H_{\kappa}^*(q)$.

³ This result implies the Sato–Tate conjecture.

Indeed, there is a slight abuse here since we only know that

$$\lim_{\substack{q \in \mathcal{P} \\ q \rightarrow +\infty}} \mu_q^h(H_\kappa^*(q)) = 1 \tag{2.16}$$

(see Remark 3.12), which means that μ_q^h is an ‘‘asymptotic’’ probability measure. If X_q is a measurable complex-valued function on $H_\kappa^*(q)$, then it is very natural to compute its *expectation* defined by

$$\mathbb{E}_q^h(X_q) := \sum_{f \in H_\kappa^*(q)}^h X_q(f),$$

its *variance* defined by

$$\mathbb{V}_q^h(X_q) := \mathbb{E}_q^h\left((X_q - \mathbb{E}_q^h(X_q))^2\right)$$

and its m -th *moments* given by

$$\mathbb{M}_{q,m}^h(X_q) := \mathbb{E}_q^h\left((X_q - \mathbb{E}_q^h(X_q))^m\right)$$

for any integer $m \geq 1$. If $X := (X_q)_{q \in \mathcal{P}}$ is a sequence of such measurable complex-valued functions, then we may legitimately wonder if the associated complex sequences

$$\left(\mathbb{E}_q^h(X_q)\right)_{q \in \mathcal{P}}, \quad \left(\mathbb{V}_q^h(X_q)\right)_{q \in \mathcal{P}}, \quad \left(\mathbb{M}_{q,m}^h(X_q)\right)_{q \in \mathcal{P}}$$

converge as q goes to infinity among the primes. If yes, the following general notations will be used for their limits:

$$\mathbb{E}_\infty^h(X), \quad \mathbb{V}_\infty^h(X), \quad \mathbb{M}_{\infty,m}^h(X)$$

for any natural integer m . In addition, these potential limits are called *asymptotic expectation*, *asymptotic variance* and *asymptotic m -th moments* of X for any natural integer $m \geq 1$.

For the end of this section, we assume that r is *odd*. We remark that the sign of the functional equations of any $L(\text{Sym}^r f, s)$ when q goes to infinity among the prime numbers and f ranges over $H_\kappa^*(q)$ is not constant as it depends on $\varepsilon_f(q)$. Let

$$H_\kappa^\varepsilon(q) := \{f \in H_\kappa^*(q) : \varepsilon(\text{Sym}^r f) = \varepsilon\}$$

where $\varepsilon = \pm 1$. If $f \in H_\kappa^{+1}(q)$, then $\text{Sym}^r f$ is said to be *even* whereas it is said to be *odd* if $f \in H_\kappa^{-1}(q)$. It is well known that

$$\lim_{\substack{q \in \mathcal{P} \\ q \rightarrow +\infty}} \mu_q^h (\{f \in H_\kappa^*(q) : \varepsilon_f(q) = \varepsilon\}) = \frac{1}{2}.$$

Since $\varepsilon(\text{Sym}^r f)$ is $\varepsilon_q(f)$ up to a sign depending only on κ and r (by Hypothesis Nice(r, f)), it follows that

$$\lim_{\substack{q \in \mathcal{P} \\ q \rightarrow +\infty}} \mu_q^h (H_\kappa^\varepsilon(q)) = \frac{1}{2}. \tag{2.17}$$

For X_q as previous, we can compute its *signed expectation* defined by

$$\mathbb{E}_q^{h,\varepsilon} (X_q) := 2 \sum_{f \in H_\kappa^\varepsilon(q)}^h X_q(f),$$

its *signed variance* defined by

$$\mathbb{V}_q^{h,\varepsilon} (X_q) := \mathbb{E}_q^{h,\varepsilon} \left((X_q - \mathbb{E}_q^{h,\varepsilon} (X_q))^2 \right)$$

and its *signed m -th moments* given by

$$\mathbb{M}_{q,m}^{h,\varepsilon} (X_q) := \mathbb{E}_q^{h,\varepsilon} \left((X_q - \mathbb{E}_q^{h,\varepsilon} (X_q))^m \right)$$

for any natural integer $m \geq 1$. In case of existence, we write $\mathbb{E}_\infty^{h,\varepsilon} (X)$, $\mathbb{V}_\infty^{h,\varepsilon} (X)$ and $\mathbb{M}_{\infty,m}^{h,\varepsilon} (X)$ for the limits, which are called *signed asymptotic expectation*, *signed asymptotic variance* and *signed asymptotic moments*. The signed expectation and the expectation are linked through the formula

$$\begin{aligned} \mathbb{E}_q^{h,\varepsilon} (X_q) &= 2 \sum_{f \in H_\kappa^*(q)}^h \frac{1 + \varepsilon \times \varepsilon(\text{Sym}^r f)}{2} X_q(f) \\ &= \mathbb{E}_q^h (X_q) - \varepsilon \times \varepsilon(\kappa, r) \sqrt{q} \sum_{f \in H_\kappa^*(q)}^h \lambda_f(q) X_q(f). \end{aligned} \tag{2.18}$$

3 Main technical ingredients of this work

3.1 Large sieve inequalities for Kloosterman sums

One of the main ingredients of this work is some large sieve inequalities for Kloosterman sums, which were established by J.-M. Deshouillers & H. Iwaniec in [6] and then refined by V. Blomer, G. Harcos & P. Michel in [3]. The proof of these large sieve inequalities relies on the spectral theory of automorphic forms on $\text{GL}_2(\mathbb{A}_\mathbb{Q})$. In particular, the authors have to understand the size of the Fourier

coefficients of these automorphic cusp forms. We have already seen that the size of the Fourier coefficients of holomorphic cusp forms is well understood (2.3) but we only have partial results on the size of the Fourier coefficients of Maaß cusp forms, which do not come from holomorphic forms. We introduce the following hypothesis, which measures the approximation towards the *Ramanujan–Petersson–Selberg conjecture*.

Hypothesis $H_2(\theta)$. If $\pi := \bigotimes'_{p \in \mathcal{P} \cup \{\infty\}} \pi_p$ is any automorphic cuspidal form on $GL_2(\mathbb{A}_{\mathbb{Q}})$ with local Hecke parameters $\alpha_{\pi}^{(1)}(p)$, $\alpha_{\pi}^{(2)}(p)$ at any prime number p and $\mu_{\pi}^{(1)}(\infty)$, $\mu_{\pi}^{(2)}(\infty)$ at infinity, then

$$\forall j \in \{1, 2\}, \quad |\alpha_{\pi}^{(j)}(p)| \leq p^{\theta}$$

for any prime number p for which π_p is unramified and

$$\forall j \in \{1, 2\}, \quad \left| \operatorname{Re} \left(\mu_{\pi}^{(j)}(\infty) \right) \right| \leq \theta$$

provided π_{∞} is unramified.

Definition 3.1. We say that θ is *admissible* if Hypothesis $H_2(\theta)$ is satisfied.

Remark 3.2. The smallest admissible value of θ is currently $\theta_0 = \frac{7}{64}$ thanks to the works of H. Kim, F. Shahidi and P. Sarnak [23, 22]. The Ramanujan–Petersson–Selberg conjecture asserts that 0 is admissible.

Definition 3.3. Given $T: \mathbb{R}^3 \rightarrow \mathbb{R}^+$ and $(M, N, C) \in (\mathbb{R} \setminus \{0\})^3$, we say that a smooth function $h: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfies the property $P(T; M, N, C)$ if for any nonnegative integers i, j, k there exists a real number $K > 0$ such that

$$\forall (x_1, x_2, x_3) \in \left[\frac{M}{2}, 2M \right] \times \left[\frac{N}{2}, 2N \right] \times \left[\frac{C}{2}, 2C \right],$$

$$x_1^i x_2^j x_3^k \frac{\partial^{i+j+k} h}{\partial x_1^i \partial x_2^j \partial x_3^k} (x_1, x_2, x_3) \leq KT(M, N, C) \left(1 + \frac{\sqrt{MN}}{C} \right)^{i+j+k}.$$

With this definition in mind, we are able to write the following proposition, which is special case of a large sieve inequality adapted from the one of Deshouillers & Iwaniec [6, Theorem 9] by Blomer, Harcos & Michel [3, Theorem 4].

Proposition 3.4. Let q be some positive integer. Let $M, N, C \geq 1$ and g be a smooth function satisfying property $P(1; M, N, C)$. Consider two sequences of complex numbers $(a_m)_{m \in [M/2, 2M]}$ and $(b_n)_{n \in [N/2, 2N]}$. If θ is admissible

and $MN \ll C^2$, then

$$\sum_{\substack{c \geq 1 \\ q|c}} \sum_{m \geq 1} \sum_{n \geq 1} a_m b_n \frac{S(m, \pm n; c)}{c} g(m, n; c) \tag{3.1}$$

$$\ll_\varepsilon (qMNC)^\varepsilon \left(\frac{C^2}{MN}\right)^\theta \left(1 + \frac{M}{q}\right)^{1/2} \left(1 + \frac{N}{q}\right)^{1/2} \|a\|_2 \|b\|_2$$

for any $\varepsilon > 0$.

We shall use a test function. For any $\nu > 0$ let us define $\mathfrak{S}_\nu(\mathbb{R})$ as the space of even Schwartz function Φ whose Fourier transform

$$\widehat{\Phi}(\xi) := \mathcal{F}[x \mapsto \Phi(x)](\xi) := \int_{\mathbb{R}} \Phi(x)e(-x\xi) dx$$

is compactly supported in $[-\nu, +\nu]$. Thanks to the Fourier inversion formula

$$\Phi(x) = \int_{\mathbb{R}} \widehat{\Phi}(\xi)e(x\xi) dx = \mathcal{F}[\xi \mapsto \widehat{\Phi}(\xi)](-x), \tag{3.2}$$

such a function Φ can be extended to an entire even function which satisfies

$$\forall s \in \mathbb{C}, \quad \Phi(s) \ll_n \frac{\exp(\nu|\operatorname{Im} s|)}{(1 + |s|)^n} \tag{3.3}$$

for any integer $n \geq 0$. The version of the large sieve inequality we shall use several times in this paper is then the following.

Corollary 3.5. *Let q be some prime number, $k_1, k_2 > 0$ be some integers, α_1, α_2, ν be some positive real numbers and $\Phi \in \mathfrak{S}_\nu(\mathbb{R})$. Let h be some smooth function satisfying property $P(T; M, N, C)$ for any $1 \leq M \leq q^{k_1\alpha_1\nu}$, $1 \leq N \leq q^{k_2\alpha_2\nu}$ and $C \geq q$. Let*

$$(a_p)_{\substack{p \in \mathcal{P} \\ p \leq q^{\alpha_1\nu}}} \quad \text{and} \quad (b_p)_{\substack{p \in \mathcal{P} \\ p \leq q^{\alpha_2\nu}}}$$

be sequences of complex numbers. If θ is admissible and $\nu \leq 2/(k_1\alpha_1 + k_2\alpha_2)$, then

$$\sum_{\substack{c \geq 1 \\ q|c}} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid q}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid q}} \left(a_{p_1} b_{p_2} \frac{S(p_1^{k_1}, p_2^{k_2}; c)}{c} h(p_1^{k_1}, p_2^{k_2}; c) \right. \\ \left. \times \widehat{\Phi}\left(\frac{\log p_1}{\log(q^{\alpha_1})}\right) \widehat{\Phi}\left(\frac{\log p_2}{\log(q^{\alpha_2})}\right) \right) \\ \ll q^\varepsilon \sum_{\substack{\# \\ 1 \leq M \leq q^{\nu\alpha_1 k_1} \\ 1 \leq N \leq q^{\nu\alpha_2 k_2} \\ C \geq q/2}} \left(1 + \sqrt{\frac{M}{q}}\right) \left(1 + \sqrt{\frac{N}{q}}\right) \left(\frac{C^2}{MN}\right)^\theta T(M, N, C) \|a\|_2 \|b\|_2 \tag{3.4}$$

where \sharp indicates that the sum is on powers of $\sqrt{2}$. The constant implied by the symbol \ll depends at most on $\varepsilon, k_1, k_2, \alpha_1, \alpha_2$ and ν .

Proof. Define $(\widehat{a}_m)_{m \in \mathbb{N}}, (\widehat{b}_n)_{n \in \mathbb{N}}$ and $g(m, n; c)$ by

$$\widehat{a}_m := a_{m^{1/k_1}} \mathbb{1}_{\mathcal{P}^{k_1}}(m) \mathbb{1}_{[1, q^{\nu\alpha_1 k_1}]}(m), \tag{3.5}$$

$$\widehat{b}_n := b_{n^{1/k_1}} \mathbb{1}_{\mathcal{P}^{k_1}}(n) \mathbb{1}_{[1, q^{\nu\alpha_1 k_1}]}(n), \tag{3.6}$$

$$g(m, n; c) := h(m, n, c) \widehat{\Phi} \left(\frac{\log m}{\log(q^{\alpha_1 k_1})} \right) \widehat{\Phi} \left(\frac{\log n}{\log(q^{\alpha_2 k_2})} \right). \tag{3.7}$$

Using a smooth partition of unity, as detailed in Section A.1, we need to evaluate

$$\sum_{\substack{1 \leq M \leq q^{\nu\alpha_1 k_1} \\ 1 \leq N \leq q^{\nu\alpha_2 k_2} \\ C \geq q/2}}^{\sharp} T(M, N, C) \sum_{\substack{c \geq 1 \\ q | c}} \sum_{m \geq 1} \sum_{n \geq 1} \widehat{a}_m \widehat{b}_n \frac{S(m, n; c)}{c} \frac{g_{M, N, C}(m, n; c)}{T(M, N, C)}. \tag{3.8}$$

Since $\nu \leq 2/(\alpha_1 k_1 + \alpha_2 k_2)$, the first summation is restricted to $MN \ll C^2$ hence, using Proposition 3.4, the quantity in (3.8) is

$$\ll \|a\|_2 \|b\|_2 q^\varepsilon \sum_{\substack{1 \leq M \leq q^{\nu\alpha_1 k_1} \\ 1 \leq N \leq q^{\nu\alpha_2 k_2} \\ C \geq q/2}}^{\sharp} T(M, N, C) \left(1 + \sqrt{\frac{M}{q}} \right) \left(1 + \sqrt{\frac{N}{q}} \right) \left(\frac{C^2}{MN} \right)^\theta.$$

□

3.2 Riemann’s explicit formula for symmetric power L -functions

In this section, we give an analog of Riemann–von Mangoldt’s explicit formula for symmetric power L -functions. Before that, let us recall some preliminary facts on zeros of symmetric power L -functions, which can be found in Section 5.3 of [19]. Let $r \geq 1$ and $f \in H_\kappa^*(q)$ for which Hypothesis Nice(r, f) holds. All the zeros of $\Lambda(\text{Sym}^r f, s)$ are in the critical strip $\{s \in \mathbb{C} : 0 < \text{Re } s < 1\}$. The multiset of the zeros of $\Lambda(\text{Sym}^r f, s)$ counted with multiplicities is given by

$$\left\{ \rho_{f,r}^{(j)} = \beta_{f,r}^{(j)} + i \gamma_{f,r}^{(j)} : j \in \mathcal{E}(f, r) \right\}$$

where

$$\mathcal{E}(f, r) := \begin{cases} \mathbb{Z}, & \text{if } \text{Sym}^r f \text{ is odd,} \\ \mathbb{Z} \setminus \{0\}, & \text{if } \text{Sym}^r f \text{ is even,} \end{cases}$$

and

$$\beta_{f,r}^{(j)} = \operatorname{Re} \rho_{f,r}^{(j)}, \quad \gamma_{f,r}^{(j)} = \operatorname{Im} \rho_{f,r}^{(j)}$$

for any $j \in \mathcal{E}(f, r)$. We enumerate the zeros such that

- (i) the sequence $j \mapsto \gamma_{f,r}^{(j)}$ is increasing,
- (ii) we have $j \geq 0$ if and only if $\gamma_{f,r}^{(j)} \geq 0$,
- (iii) we have $\rho_{f,r}^{(-j)} = 1 - \rho_{f,r}^{(j)}$.

Note that if $\rho_{f,r}^{(j)}$ is a zero of $\Lambda(\operatorname{Sym}^r f, s)$, then $\overline{\rho_{f,r}^{(j)}}$, $1 - \rho_{f,r}^{(j)}$ and $1 - \overline{\rho_{f,r}^{(j)}}$ are also zeros of $\Lambda(\operatorname{Sym}^r f, s)$. In addition, remember that if $\operatorname{Sym}^r f$ is odd, then the functional equation of $L(\operatorname{Sym}^r f, s)$ evaluated at the critical point $s = 1/2$ provides a trivial zero denoted by $\rho_{f,r}^{(0)}$. It can be shown [19, Theorem 5.8] that the number of zeros $\Lambda(\operatorname{Sym}^r f, s)$ satisfying $|\gamma_{f,r}^{(j)}| \leq T$ is

$$\frac{T}{\pi} \log \left(\frac{q^r T^{r+1}}{(2\pi e)^{r+1}} \right) + O(\log(qT)) \tag{3.9}$$

as $T \geq 1$ goes to infinity. We state now the *Generalised Riemann Hypothesis*, which is the main conjecture about the horizontal distribution of the zeros of $\Lambda(\operatorname{Sym}^r f, s)$ in the critical strip.

Hypothesis GRH(r). For any prime number q and any f in $H_\kappa^*(q)$, all the zeros of $\Lambda(\operatorname{Sym}^r f, s)$ lie on the critical line $\{s \in \mathbb{C} : \operatorname{Re} s = 1/2\}$, namely $\beta_{r,f}^{(j)} = 1/2$ for any $j \in \mathcal{E}(f, r)$.

Remark 3.6. We do not use this hypothesis in our proofs.

Under Hypothesis GRH(r) it can be shown that the number of zeros of the function $\Lambda(\operatorname{Sym}^r f, s)$ satisfying $|\gamma_{f,r}^{(j)}| \leq 1$ is given by

$$\frac{1}{\pi} \log(q^r)(1 + o(1))$$

as q goes to infinity. Thus, the spacing between two consecutive zeros with imaginary part in $[0, 1]$ is roughly of size

$$\frac{2\pi}{\log(q^r)}. \tag{3.10}$$

We aim at studying the local distribution of the zeros of $\Lambda(\operatorname{Sym}^r f, s)$ in a neighborhood of the real axis of size $1/\log q^r$ since in such a neighborhood, we expect to catch only a few zeros (but without being able to say that we catch only one⁴).

⁴ We refer to Miller [27] and Omar [29] for works related to the “first” zero.

Hence, we normalise the zeros by defining

$$\widehat{\rho}_{f,r}^{(j)} := \frac{\log(q^r)}{2i\pi} \left(\beta_{f,r}^{(j)} - \frac{1}{2} + i\gamma_{f,r}^{(j)} \right).$$

Note that

$$\widehat{\rho}_{f,r}^{(-j)} = -\widehat{\rho}_{f,r}^{(j)}.$$

Definition 3.7. Consider $f \in H_{\kappa}^*(q)$ for which Hypothesis Nice(r, f) holds and let $\Phi \in \mathcal{S}_\nu(\mathbb{R})$. The *one-level density* (relatively to Φ) of $\text{Sym}^r f$ is

$$D_{1,q}[\Phi; r](f) := \sum_{j \in \mathcal{E}(f,r)} \Phi \left(\widehat{\rho}_{f,r}^{(j)} \right). \tag{3.11}$$

To study $D_{1,q}[\Phi; r](f)$ for any $\Phi \in \mathcal{S}_\nu(\mathbb{R})$, we transform this sum over zeros into a sum over primes in the next proposition. In other words, we establish an explicit formula for symmetric power L -functions. Since the proof is classical, we refer to [20, §4] or [10, §2.2], where one can find a method that has just to be adapted to our setting.

Proposition 3.8. Let $r \geq 1$, $f \in H_{\kappa}^*(q)$ satisfying Hypothesis Nice(r, f) and let $\Phi \in \mathcal{S}_\nu(\mathbb{R})$. We have

$$\begin{aligned} D_{1,q}[\Phi; r](f) &= E[\Phi; r] + P_q^1[\Phi; r](f) \\ &\quad + \sum_{m=0}^{r-1} (-1)^m P_q^2[\Phi; r, m](f) + O\left(\frac{1}{\log(q^r)}\right) \end{aligned}$$

where

$$\begin{aligned} E[\Phi; r] &:= \widehat{\Phi}(0) + \frac{(-1)^{r+1}}{2} \Phi(0), \\ P_q^1[\Phi; r](f) &:= -\frac{2}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \lambda_f(p^r) \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right), \\ P_q^2[\Phi; r, m](f) &:= -\frac{2}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \lambda_f(p^{2(r-m)}) \frac{\log p}{p} \widehat{\Phi}\left(\frac{2 \log p}{\log(q^r)}\right) \end{aligned}$$

for any integer $m \in \{0, \dots, r-1\}$.

3.3 Contribution of the old forms

In this short section, we prove the following useful lemmas.

Lemma 3.9. *Let $p_1, p_2 \neq q$ be some prime numbers and a_1, a_2, a be some non-negative integers. Then*

$$\sum_{\ell|q^\infty} \frac{\Delta_1(\ell^2 p_1^{a_1}, p_2^{a_2} q^a)}{\ell} \ll \frac{1}{q^{a/2}},$$

the implied constant depending only on a_1 and a_2 .

Proof. Using Proposition 2.1 and the fact that $H_\kappa(1) = H_\kappa^*(1)$, we write

$$\Delta_1(\ell^2 p_1^{a_1}, p_2^{a_2} q^a) = \sum_{f \in H_\kappa^*(1)}^h \lambda_f(\ell^2 p_1^{a_1}) \lambda_f(p_2^{a_2} q^a) \tag{3.12}$$

$$\ll \sum_{f \in H_\kappa^*(1)}^h |\lambda_f(\ell^2 p_1^{a_1})| \cdot |\lambda_f(p_2^{a_2})| \cdot |\lambda_f(q^a)|. \tag{3.13}$$

By Deligne’s bound (2.3) we have

$$|\lambda_f(\ell^2 p_1^{a_1})| \cdot |\lambda_f(p_2^{a_2})| \leq \tau(\ell^2 p_1^{a_1}) \tau(p_2^{a_2}) \leq (a_1 + 1)(a_2 + 1) \tau(\ell^2). \tag{3.14}$$

By the multiplicativity relation (2.4) and the value of the sign of the functional equation (2.15), we have

$$|\lambda_f(q^a)| \ll \frac{1}{q^{a/2}}. \tag{3.15}$$

We obtain the result according to the estimates (3.15), (3.14) and equation (3.13) and by using (2.16) and

$$\sum_{\ell|q^\infty} \frac{\tau(\ell^2)}{\ell} = \frac{1 + 1/q}{(1 - 1/q)^2} \ll 1. \tag{3.16} \quad \square$$

Lemma 3.10. *Let $m, n \geq 1$ be some coprime integers. Then*

$$\Delta_q(m, n) - \delta(m, n) \ll \begin{cases} \frac{(mn)^{1/4}}{q} \log\left(\frac{mn}{q^2}\right), & \text{if } mn > q^2, \\ \frac{(mn)^{(\kappa-1)/2}}{q^{\kappa-1/2}} \leq \frac{(mn)^{1/4}}{q}, & \text{if } mn \leq q^2. \end{cases}$$

Proof. This is a direct consequence of the Weil–Estermann bound (A.6) and Lemma A.1. □

Corollary 3.11. *For any prime number q , we have*

$$\sqrt{q} \sum_{f \in H_{\kappa}^*(q)}^h \lambda_f(q) \ll \frac{1}{q^{\delta_{\kappa}}}$$

where

$$\delta_{\kappa} := \begin{cases} \frac{\kappa}{2} - 1, & \text{if } \kappa \leq 10 \text{ or } \kappa = 14, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Let $\mathcal{K} = \{\kappa \in 2\mathbb{N} : 2 \leq \kappa \leq 14, \kappa \neq 12\}$. By Proposition 2.2, we have

$$\sum_{f \in H_{\kappa}^*(q)}^h \lambda_f(q) = \Delta_q(1, q) - \frac{\delta(\kappa \notin \mathcal{K})}{q^{\nu(q)}} \sum_{\ell | q^{\infty}} \frac{\Delta_1(\ell^2, q)}{\ell}. \tag{3.16}$$

The term $\delta(\kappa \notin \mathcal{K})$ comes from Proposition 2.1 and the fact that there are no cusp forms of weight $\kappa \in \mathcal{K}$ and level 1. Lemma 3.10 gives

$$\Delta_q(1, q) \ll \frac{1}{q^{\kappa/2}} \tag{3.17}$$

and Lemma 3.9 gives

$$\sum_{\ell | q^{\infty}} \frac{\Delta_1(\ell^2, q)}{\ell} \ll \frac{1}{\sqrt{q}}. \tag{3.18}$$

Since $\nu(q) > q$, the result follows from the estimates (3.17) and (3.18) and the equation (3.16). □

Remark 3.12. In a very similar fashion, one can prove that

$$\mu_q^h(H_{\kappa}^*(q)) = \mathbb{E}_q^h(1) = 1 + O\left(\frac{1}{q^{\gamma_{\kappa}}}\right) \tag{3.19}$$

where

$$\gamma_{\kappa} := \begin{cases} \kappa - \frac{1}{2}, & \text{if } \kappa \leq 10 \text{ or } \kappa = 14, \\ 1, & \text{otherwise.} \end{cases}$$

Corollary 3.11, (3.19) and (2.18) imply

$$\mathbb{E}_q^{h,\varepsilon}(1) = 1 + O\left(\frac{1}{q^{\beta_{\kappa}}}\right) \tag{3.20}$$

where

$$\beta_{\kappa} := \begin{cases} \frac{\kappa}{2} - 1, & \text{if } \kappa \leq 10 \text{ or } \kappa = 14, \\ 1, & \text{otherwise.} \end{cases}$$

A direct consequence of Lemma 3.9 is the following one.

Lemma 3.13. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, w$ be some nonnegative real numbers. Let Φ_1 and Φ_2 be in $\mathcal{S}_\nu(\mathbb{R})$. Then*

$$\sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid q}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid q}} \frac{\log p_1}{p_1^{\alpha_1}} \frac{\log p_2}{p_2^{\alpha_2}} \widehat{\Phi}_1\left(\frac{\log p_1}{\log(q^{\beta_1})}\right) \widehat{\Phi}_2\left(\frac{\log p_2}{\log(q^{\beta_2})}\right) \sum_{\ell | q^\infty} \frac{\Delta_1(\ell^2 p_1^{\gamma_1}, p_2^{\gamma_2} q^w)}{\ell} \ll q^{\delta\nu - w/2 + \varepsilon}$$

with δ given in Table 1.

$\alpha_1 \backslash \alpha_2$	$]0, 1]$	$[1, +\infty[$
$]0, 1]$	$\beta_1(1 - \alpha_1) + \beta_2(1 - \alpha_2)$	$\beta_2(1 - \alpha_2)$
$[1, +\infty[$	$\beta_1(1 - \alpha_1)$	0

Table 1. Values of δ .

4 Linear statistics for low-lying zeros

4.1 Density results for families of L -functions

We briefly recall some well-known features that can be found in [20]. Let \mathcal{F} be a family of L -functions indexed by the arithmetic conductor, namely

$$\mathcal{F} = \bigcup_{Q \geq 1} \mathcal{F}(Q)$$

where the arithmetic conductor of any L -function in $\mathcal{F}(Q)$ is of order Q in the logarithmic scale. It is expected that there is a symmetry group $G(\mathcal{F})$ of matrices of large rank endowed with a probability measure, which can be associated to \mathcal{F} such that the low-lying zeros of the L -functions in \mathcal{F} , namely the non-trivial zeros of height less than $1/\log Q$, are distributed like the eigenvalues of the matrices in $G(\mathcal{F})$. In other words, there should exist a symmetry group $G(\mathcal{F})$ such that for any $\nu > 0$ and any $\Phi \in \mathcal{S}_\nu(\mathbb{R})$,

$$\begin{aligned} \lim_{Q \rightarrow +\infty} \frac{1}{\mathcal{F}(Q)} \sum_{\pi \in \mathcal{F}(Q)} \sum_{\substack{0 \leq \beta_\pi \leq 1 \\ \gamma_\pi \in \mathbb{R} \\ L(\pi, \beta_\pi + i\gamma_\pi) = 0}} \Phi\left(\frac{\log Q}{2i\pi} \left(\beta_\pi - \frac{1}{2} + i\gamma_\pi\right)\right) \\ = \int_{\mathbb{R}} \Phi(x) W_1(G(\mathcal{F}))(x) dx \end{aligned} \tag{4.1}$$

where $W_1(G(\mathcal{F}))$ is the one-level density of the eigenvalues of $G(\mathcal{F})$. In this case, \mathcal{F} is said to be of *symmetry type* $G(\mathcal{F})$. Proving a *density result* for \mathcal{F} is proving (4.1) for some $\nu > 0$. For instance, the following densities are determined in [21]:

$$\begin{aligned} W_1(\text{SO}(\text{even}))(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x}, \\ W_1(\text{O})(x) &= 1 + \frac{1}{2}\delta_0(x), \\ W_1(\text{SO}(\text{odd}))(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta_0(x), \\ W_1(\text{Sp})(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} \end{aligned}$$

where δ_0 is the Dirac distribution at 0. According to Plancherel formula

$$\int_{\mathbb{R}} \Phi(x)W_1(G(\mathcal{F}))(x) dx = \int_{\mathbb{R}} \widehat{\Phi}(x)\widehat{W}_1(G(\mathcal{F}))(x) dx$$

and one can check that

$$\begin{aligned} \widehat{W}_1(\text{SO}(\text{even}))(x) &= \delta_0(x) + \frac{1}{2}\eta(x), \\ \widehat{W}_1(\text{O})(x) &= \delta_0(x) + \frac{1}{2}, \\ \widehat{W}_1(\text{SO}(\text{odd}))(x) &= \delta_0(x) - \frac{1}{2}\eta(x) + 1, \\ \widehat{W}_1(\text{Sp})(x) &= \delta_0(x) - \frac{1}{2}\eta(x) \end{aligned}$$

where

$$\eta(x) := \begin{cases} 1, & \text{if } |x| < 1, \\ \frac{1}{2} & \text{if } x = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, if one can only prove a density result for $\nu \leq 1$, the three orthogonal densities are indistinguishable although they are distinguishable from Sp. Thus, the analytic challenge is to pass the natural barrier $\nu = 1$.

4.2 Asymptotic expectation of the one-level density

The aim of this part is to prove a density result for the family

$$\mathcal{F}_r := \bigcup_{q \in \mathcal{P}} \{L(\text{Sym}^r f, s) : f \in H_\kappa^*(q)\}$$

for any $r \geq 1$. This requires proving the existence and computing the asymptotic expectation $\mathbb{E}_\infty^h(D_1[\Phi; r])$ of $D_1[\Phi; r] := (D_{1,q}[\Phi; r])_{q \in \mathcal{P}}$ for any $r \geq 1$ and for Φ in $\mathcal{S}_\nu(\mathbb{R})$ with $\nu > 0$ as large as possible in order to be able to distinguish between the three orthogonal densities if r is small enough. Recall that $E[\Phi; r]$ has been defined in Proposition 3.8.

Theorem 4.1. *Let $r \geq 1$ and $\Phi \in \mathcal{S}_\nu(\mathbb{R})$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any $f \in H_\kappa^*(q)$ and also that θ is admissible. Let*

$$\nu_{1,\max}(r, \kappa, \theta) := \left(1 - \frac{1}{2(\kappa - 2\theta)}\right) \frac{2}{r^2}.$$

If $\nu < \nu_{1,\max}(r, \kappa, \theta)$, then

$$\mathbb{E}_\infty^h(D_1[\Phi; r]) = E[\Phi; r].$$

Remark 4.2. We remark that

$$\nu_{1,\max}(r, \kappa, \theta_0) = \left(1 - \frac{16}{32\kappa - 7}\right) \frac{2}{r^2} \geq \frac{82}{57r^2}, \tag{4.2}$$

$$\nu_{1,\max}(r, \kappa, 0) = \left(1 - \frac{1}{2\kappa}\right) \frac{2}{r^2} \geq \frac{3}{2r^2} \tag{4.3}$$

and thus $\nu_{1,\max}(1, \kappa, \theta_0) > 1$ whereas $\nu_{1,\max}(r, \kappa, \theta_0) \leq 1$ for any $r \geq 2$.

Remark 4.3. Note that

$$E[\Phi; r] = \int_{\mathbb{R}} \widehat{\Phi}(x) \left(\delta_0(x) + \frac{(-1)^{r+1}}{2}\right) dx.$$

Thus, this theorem reveals that the symmetry type of \mathcal{F}_r is

$$G(\mathcal{F}_r) = \begin{cases} \text{Sp}, & \text{if } r \text{ is even,} \\ \text{O}, & \text{if } r = 1, \\ \text{SO(even) or O or SO(odd),} & \text{if } r \geq 3 \text{ is odd.} \end{cases}$$

Some additional comments are given in Remark 1.4 on page 973.

Proof of Theorem 4.1. The proof is detailed and will be a model for the next density results. According to Proposition 3.8 and (3.19), we have

$$\begin{aligned} \mathbb{E}_q^h(D_{1,q}[\Phi; r]) &= E[\Phi; r] + \mathbb{E}_q^h(P_q^1[\Phi; r]) \\ &\quad + \sum_{m=0}^{r-1} (-1)^m \mathbb{E}_q^h(P_q^2[\Phi; r, m]) + O\left(\frac{1}{\log(q^r)}\right). \end{aligned} \tag{4.4}$$

The first term in (4.4) is the main term given in the theorem. We now estimate the second term of (4.4) *via* the trace formula given in Proposition 2.2:

$$\mathbb{E}_q^h(P_q^1[\Phi; r]) = \mathbb{P}_{q,\text{new}}^1[\Phi; r] + \mathbb{P}_{q,\text{old}}^1[\Phi; r] \tag{4.5}$$

where

$$\begin{aligned} \mathbb{P}_{q,\text{new}}^1[\Phi; r] &= -\frac{2}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \Delta_q(p^r, 1) \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right), \\ \mathbb{P}_{q,\text{old}}^1[\Phi; r] &= \frac{2}{q \log(q^r)} \sum_{\ell \mid q^\infty} \frac{1}{\ell} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \Delta_1(p^r \ell^2, 1) \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right). \end{aligned}$$

Let us estimate the new part, which can be written as

$$\begin{aligned} \mathbb{P}_{q,\text{new}}^1[\Phi; r] &= -\frac{2(2\pi i^\kappa)}{\log(q^r)} \sum_{\substack{c \geq 1 \\ q \mid c}} \sum_{p \in \mathcal{P}} \left(\frac{\log p}{\sqrt{p}} \delta_{q \nmid p} \mathbb{1}_{[1, q^{rv}]}(p)\right) \frac{S(p^r, 1; c)}{c} \\ &\quad \times J_{\kappa-1}\left(\frac{4\pi \sqrt{p^r}}{c}\right) \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right). \end{aligned}$$

Thanks to (A.3), the function

$$h(m; c) := J_{\kappa-1}\left(\frac{4\pi \sqrt{m}}{c}\right)$$

satisfies Hypothesis $\mathbf{P}(T; M, 1, C)$ with

$$T(M, 1, C) = \left(1 + \frac{\sqrt{M}}{C}\right)^{1/2-\kappa} \left(\frac{\sqrt{M}}{C}\right)^{\kappa-1}.$$

Hence, if $\nu \leq 2/r^2$, then Corollary 3.5 leads to

$$\mathbb{P}_{q,\text{new}}^1[\Phi; r] \ll_\varepsilon q^\varepsilon \sum_{\substack{1 \leq M \leq q^{\nu r^2} \\ C \geq q/2}}^\# \left(1 + \sqrt{\frac{M}{q}}\right) \left(\frac{\sqrt{M}}{C}\right)^{\kappa-1-2\theta} \tag{4.6}$$

$$\ll_\varepsilon q^\varepsilon \sum_{1 \leq M \leq q^{\nu r^2}}^\# \left(\frac{M^{\frac{\kappa-1}{2}-\theta}}{q^{\kappa-1-2\theta}} + \frac{M^{\frac{\kappa}{2}-\theta}}{q^{\kappa-\frac{1}{2}-2\theta}}\right) \tag{4.7}$$

thanks to (A.2). Summing over M via (A.1) leads to

$$\mathbb{P}_{q,\text{new}}^1[\Phi; r] \ll_\varepsilon q^{(\frac{\kappa-1}{2}-\theta)(r^2\nu-2)+\varepsilon} + q^{(\frac{\kappa}{2}-\theta)r^2\nu-(\kappa-\frac{1}{2}-2\theta)+\varepsilon}, \tag{4.8}$$

which is an admissible error term if $\nu < \nu_{1,\text{max}}(r, \kappa, \theta)$. According to Lemma 3.13 (with $\alpha_2 = +\infty$) we have

$$\mathbb{P}_{q,\text{old}}^1[\Phi; r] \ll_\varepsilon q^{\frac{r\nu}{2}-1+\varepsilon}, \tag{4.9}$$

which is an admissible error term if $\nu < 2/r$. According to the estimates (4.8) and (4.9), we obtain from equation (4.5)

$$\mathbb{E}_q^h(P_q^1[\Phi; r]) \ll \frac{1}{q^{\delta_1}} \tag{4.10}$$

for some $\delta_1 > 0$ (depending on ν and r) when $\nu < \nu_{1,\text{max}}(r, \kappa, \theta)$. We now estimate the third term of (4.4). If $0 \leq m \leq r - 1$, then the trace formula given in Proposition 2.2 implies that

$$\mathbb{E}_q^h(P_q^2[\Phi; r, m]) = \mathbb{P}_{q,\text{new}}^2[\Phi; r, m] + \mathbb{P}_{q,\text{old}}^2[\Phi; r, m] \tag{4.11}$$

where

$$\begin{aligned} \mathbb{P}_{q,\text{new}}^2[\Phi; r, m] &= -\frac{2}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \Delta_q(p^{2(r-m)}, 1) \frac{\log p}{p} \widehat{\Phi}\left(\frac{\log(p^2)}{\log(q^r)}\right), \\ \mathbb{P}_{q,\text{old}}^2[\Phi; r, m] &= \frac{2}{q \log(q^r)} \sum_{\ell | q^\infty} \frac{1}{\ell} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \Delta_1(p^{2(r-m)}\ell^2, 1) \frac{\log p}{p} \widehat{\Phi}\left(\frac{\log(p^2)}{\log(q^r)}\right). \end{aligned}$$

Let us estimate the new part, which can be written as

$$\begin{aligned} \mathbb{P}_{q,\text{new}}^2[\Phi; r, m] &= -\frac{2(2\pi i^\kappa)}{\log(q^r)} \sum_{\substack{c \geq 1 \\ q | c}} \sum_{p \in \mathcal{P}} \left(\frac{\log p}{\sqrt{p}} \delta_{q \nmid p} \mathbb{1}_{\left[1, q^{\frac{r\nu}{2}}\right]}(p) \right) \frac{S(p^{2(r-m)}, 1; c)}{c} \\ &\quad \times \frac{1}{\sqrt{p}} J_{\kappa-1}\left(\frac{4\pi \sqrt{p^{2(r-m)}}}{c}\right) \widehat{\Phi}\left(\frac{\log p}{\log q^{r/2}}\right). \end{aligned}$$

The function

$$h(d, c) := J_{\kappa-1}\left(\frac{4\pi \sqrt{d}}{c}\right) \times \frac{1}{d^{1/(4(r-m))}}$$

satisfies Hypothesis $P(T; M, 1, C)$ with

$$T(M, 1, C) = \left(1 + \frac{\sqrt{M}}{C}\right)^{1/2-\kappa} \left(\frac{\sqrt{M}}{C}\right)^{\kappa-1} \frac{1}{M^{1/(4(r-m))}}.$$

Hence, if $\nu \leq 2/r^2$, then Corollary 3.5 leads to

$$\mathbb{P}_{q,\text{new}}^2[\Phi; r, m] \ll_{\varepsilon} q^{\varepsilon} \sum_{\substack{M \leq q^{\nu r(r-m)} \\ C \geq q/2}}^{\#} \frac{1}{(M)^{1/(4r-4m)}} \left(\frac{\sqrt{M}}{C}\right)^{\kappa-1-2\theta} \left(1 + \sqrt{\frac{M}{q}}\right).$$

This is smaller than the bound given in (4.6) and hence is an admissible error term if $\nu < \nu_{1,\max}(r, \kappa, \theta)$. According to Lemma 3.13, we have

$$\mathbb{P}_{q,\text{old}}^2[\Phi; r] \ll_{\varepsilon} q^{-1+\varepsilon}. \tag{4.12}$$

We obtain

$$\mathbb{E}_q^h(P_q^2[\Phi; r, m]) \ll \frac{1}{q^{\delta_2}} \tag{4.13}$$

for some $\delta_2 > 0$ (depending on ν and r) when $\nu < \nu_{1,\max}(r, \kappa, \theta)$. Finally, substituting (4.13) and (4.10) in (4.4), we get

$$\mathbb{E}_q^h(D_{1,q}[\Phi; r]) = E[\Phi; r] + O\left(\frac{1}{\log q}\right). \tag{4.14}$$

□

4.3 Signed asymptotic expectation of the one-level density

In this part, we prove some density results for subfamilies of \mathcal{F}_r where the sign of the functional equation is constant. The two subfamilies are defined by

$$\mathcal{F}_r^{\varepsilon} := \bigcup_{q \in \mathcal{P}} \{L(\text{Sym}^r f, s) : f \in H_{\kappa}^{\varepsilon}(q)\}.$$

Indeed, we compute the asymptotic expectation $\mathbb{E}_{\infty}^{\text{h},\varepsilon}(D_1[\Phi; r])$.

Theorem 4.4. *Let $r \geq 1$ be an odd integer, $\varepsilon = \pm 1$ and $\Phi \in \mathfrak{S}_{\nu}(\mathbb{R})$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any $f \in H_{\kappa}^{\varepsilon}(q)$ and also that θ is admissible. Let*

$$\nu_{1,\max}^{\varepsilon}(r, \kappa, \theta) := \inf\left(\nu_{1,\max}(r, \kappa, \theta), \frac{3}{r(r+2)}\right).$$

If $\nu < \nu_{1,\max}^{\varepsilon}(r, \kappa, \theta)$, then

$$\mathbb{E}_{\infty}^{\text{h},\varepsilon}(D_1[\Phi; r]) = E[\Phi; r].$$

Some comments are given in Remark 1.5 on page 973.

Proof of Theorem 4.4. By (2.18), we have

$$\mathbb{E}_q^{\text{h},\varepsilon}(D_{1,q}[\Phi; r]) = \mathbb{E}_q^{\text{h}}(D_{1,q}[\Phi; r]) - \varepsilon \times \varepsilon(k, r) \sqrt{q} \mathbb{E}_q^{\text{h}}(\lambda.(q)D_{1,q}[\Phi; r]). \quad (4.15)$$

The first term is the main term of the theorem thanks to Theorem 4.1. According to Proposition 3.8 and Corollary 3.11, the second term (without the epsilon factors) is given by

$$\begin{aligned} & \sqrt{q} \mathbb{E}_q^{\text{h}}(\lambda.(q)P_q^1[\Phi; r]) \\ & + \sqrt{q} \sum_{m=0}^{r-1} (-1)^m \mathbb{E}_q^{\text{h}}(\lambda.(q)P_q^2[\Phi; r, m]) + O\left(\frac{1}{\log(q^r)}\right). \end{aligned} \quad (4.16)$$

Let us focus on the first term in (4.16); the same discussion holds for the second term with even better results on ν . We have

$$\sqrt{q} \mathbb{E}_q^{\text{h}}(\lambda.(q)P_q^1[\Phi; r]) = \sqrt{q} \mathbb{P}_{q,\text{new}}^1[\Phi; r] + \sqrt{q} \mathbb{P}_{q,\text{old}}^1[\Phi; r] \quad (4.17)$$

where

$$\begin{aligned} \mathbb{P}_{q,\text{new}}^1[\Phi; r] &= -\frac{2}{\log(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \Delta_q(p^r q, 1) \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right), \\ \mathbb{P}_{q,\text{old}}^1[\Phi; r] &= \frac{2}{q\nu(q) \log(q^r)} \sum_{\ell | q^\infty} \frac{1}{\ell} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \Delta_1(p^r \ell^2, q) \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log(q^r)}\right). \end{aligned}$$

Lemma 3.13 implies

$$\sqrt{q} \mathbb{P}_{q,\text{old}}^1[\Phi; r] \ll q^{(\nu r - 4)/2}, \quad (4.18)$$

which is an admissible error term if $\nu < 4/r$. The new part is given by

$$\begin{aligned} \mathbb{P}_{q,\text{new}}^1[\Phi; r] &= -\frac{2(2\pi i^\kappa)}{\log(q^r)} \sum_{\substack{c \geq 1 \\ q | c}} \sum_{\substack{p \in \mathcal{P} \\ q \nmid p}} \frac{\log p}{\sqrt{p}} \frac{S(p^r q, 1; c)}{c} \\ & \quad \times J_{\kappa-1}\left(\frac{4\pi \sqrt{p^r q}}{c}\right) \widehat{\Phi}\left(\frac{\log(p)}{\log(q^r)}\right) \end{aligned}$$

and can be written as

$$-\frac{2(2\pi i^\kappa)}{\log(q^r)} \sum_{\substack{c \geq 1 \\ q | c}} \sum_{m \geq 1} \widehat{a}_m \frac{S(m, 1; c)}{c} J_{\kappa-1}\left(\frac{4\pi \sqrt{m}}{c}\right) \widehat{\Phi}\left(\frac{\log(m/q)}{\log(q^r)}\right)$$

where

$$\widehat{a}_m := \mathbb{1}_{[1, q^{1+\nu r^2}]} \begin{cases} 0, & \text{if } q \nmid m \text{ or } m \neq p^r q \text{ for some } p \neq q \text{ in } \mathcal{P}, \\ \frac{\log p}{\sqrt{p}}, & \text{if } m = p^r q \text{ for some } p \neq q \text{ in } \mathcal{P}. \end{cases}$$

Thus, if $\nu \leq 1/r^2$, then we obtain

$$\mathbb{P}_{q, \text{new}}^1[\Phi; r, m] \ll_\varepsilon q^\varepsilon \sum_{\substack{M \leq q^{1+\nu r^2} \\ C \geq q/2}}^\# \left(\frac{\sqrt{M}}{C}\right)^{\kappa-1-2\theta} \left(1 + \sqrt{\frac{M}{q}}\right)$$

as in the proof of Corollary 3.5. Summing over C via (A.2) gives

$$\mathbb{P}_{q, \text{new}}^1[\Phi; r, m] \ll_\varepsilon q^\varepsilon \sum_{M \leq q^{1+r^2\nu}}^\# \left(\frac{M^{\frac{\kappa-1}{2}-\theta}}{q^{\kappa-1-2\theta}} + \frac{M^{\frac{\kappa}{2}-\theta}}{q^{\kappa-\frac{1}{2}-2\theta}}\right).$$

Summing over M via (A.1) leads to

$$\mathbb{P}_{q, \text{new}}^1[\Phi; r, m] \ll_\varepsilon q^{(\frac{\kappa-1}{2}-\theta)r^2\nu - (\frac{\kappa-1}{2}-\theta)+\varepsilon} + q^{(\frac{\kappa}{2}-\theta)r^2\nu - (\frac{\kappa-1}{2}-\theta)+\varepsilon}, \quad (4.19)$$

which is an admissible error term if $\nu < \frac{1}{r^2} \left(1 - \frac{1}{\kappa-2\theta}\right)$. □

5 Quadratic statistics for low-lying zeros

5.1 Asymptotic expectation of the two-level density and asymptotic variance

Definition 5.1. Let $f \in H_\kappa^*(q)$ and Φ_1, Φ_2 in $\mathcal{S}_\nu(\mathbb{R})$. The *two-level density* (relatively to Φ_1 and Φ_2) of $\text{Sym}^r f$ is

$$D_{2,q}[\Phi_1, \Phi_2; r](f) := \sum_{\substack{(j_1, j_2) \in \mathcal{E}(f, r)^2 \\ j_1 \neq \pm j_2}} \Phi_1\left(\widehat{\rho}_{f,r}^{(j_1)}\right) \Phi_2\left(\widehat{\rho}_{f,r}^{(j_2)}\right).$$

Remark 5.2. In this definition, it is important to note that the condition $j_1 \neq j_2$ does not imply that $\widehat{\rho}_{f,r}^{(j_1)} \neq \widehat{\rho}_{f,r}^{(j_2)}$. It only implies this if the zeros are simple. Recall however that some L -functions of elliptic curves (hence of modular forms) have multiple zeros at the critical point [26, 4].

The following lemma is an immediate consequence of Definition 5.1.

Lemma 5.3. Let $f \in H_\kappa^*(q)$ and Φ_1, Φ_2 in $\mathcal{S}_\nu(\mathbb{R})$. Then

$$D_{2,q}[\Phi_1, \Phi_2; r](f) = D_{1,q}[\Phi_1; r](f)D_{1,q}[\Phi_2; r](f) - 2D_{1,q}[\Phi_1\Phi_2; r](f) + \mathbb{1}_{H_\kappa^{-1}(q)}(f) \times \Phi_1(0)\Phi_2(0).$$

We first evaluate the product of one-level statistics on average.

Lemma 5.4. *Let $r \geq 1$. Let Φ_1 and Φ_2 in $\mathcal{S}_\nu(\mathbb{R})$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any $f \in H_\kappa^*(q)$ and also that θ is admissible. If $\nu < 1/r^2$, then*

$$\mathbb{E}_\infty^h(D_1[\Phi_1; r]D_1[\Phi_2; r]) = E[\Phi_1; r]E[\Phi_2; r] + 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) du.$$

Remark 5.5. Since Theorem 4.1 implies that

$$\begin{aligned} &\mathbb{E}_\infty^h(D_1[\Phi_1; r]D_1[\Phi_2; r]) - E[\Phi_1; r]E[\Phi_2; r] \\ &= \mathbb{E}_\infty^h(D_1[\Phi_1; r]D_1[\Phi_2; r]) - \mathbb{E}_\infty^h(D_1[\Phi_1; r])\mathbb{E}_\infty^h(D_1[\Phi_2; r]), \end{aligned}$$

Lemma 5.4 reveals that the term

$$\mathbb{C}_\infty^h(D_1[\Phi_1; r], D_1[\Phi_2; r]) := 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) du$$

measures the dependence between $D_1[\Phi_1; r]$ and $D_1[\Phi_2; r]$. This term is the asymptotic covariance of $D_1[\Phi_1; r]$ and $D_1[\Phi_2; r]$. In particular, taking $\Phi_1 = \Phi_2$, we obtain the asymptotic variance.

Theorem 5.6. *Let $\Phi \in \mathcal{S}_\nu(\mathbb{R})$. If $\nu < 1/r^2$, then the asymptotic variance of the random variable $D_{1,q}[\Phi; r]$ is*

$$\mathbb{V}_\infty^h(D_1[\Phi; r]) = 2 \int_{\mathbb{R}} |u| \widehat{\Phi}^2(u) du.$$

Proof of Lemma 5.4. From Proposition 3.8, we obtain

$$\begin{aligned} &\mathbb{E}_q^h(D_{1,q}[\Phi_1; r]D_{1,q}[\Phi_2; r]) \\ &= E[\Phi_1; r]E[\Phi_2; r] + \mathbb{C}_q^h \\ &+ \sum_{\substack{(i,j) \in \{1,2\}^2 \\ i \neq j}} \sum_{m=0}^{r-1} (-1)^m \mathbb{E}_q^h(P_q^1[\Phi_i; r]P_q^2[\Phi_j; r, m]) \\ &+ \sum_{m_1=0}^{r-1} \sum_{m_2=0}^{r-1} (-1)^{m_1+m_2} \mathbb{E}_q^h(P_q^2[\Phi_1; r, m_1]P_q^2[\Phi_2; r, m_2]) \\ &+ O\left(\frac{1}{\log(q^r)}\right) \end{aligned} \tag{5.1}$$

with

$$\mathbb{C}_q^h := \mathbb{E}_q^h (P_q^1[\Phi_1; r] P_q^1[\Phi_2; r]).$$

The error term is evaluated by use of Theorem 4.1 and equations (2.16), (4.10) and (4.13). We first compute \mathbb{C}_q^h . Using Proposition 2.2, we compute $\mathbb{C}_q^h = E^n - 4E^o$ with

$$E^n := \frac{4}{\log^2(q^r)} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid q}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid q}} \frac{\log p_1}{\sqrt{p_1}} \frac{\log p_2}{\sqrt{p_2}} \widehat{\Phi}_1 \left(\frac{\log p_1}{\log(q^r)} \right) \times \widehat{\Phi}_2 \left(\frac{\log p_2}{\log(q^r)} \right) \Delta_q(p_1^r, p_2^r)$$

and

$$E^o := \frac{1}{q \log^2(q^r)} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid q}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid q}} \frac{\log p_1}{\sqrt{p_1}} \frac{\log p_2}{\sqrt{p_2}} \widehat{\Phi}_1 \left(\frac{\log p_1}{\log(q^r)} \right) \times \widehat{\Phi}_2 \left(\frac{\log p_2}{\log(q^r)} \right) \sum_{\ell \mid q^\infty} \frac{\Delta_1(\ell^2 p_1^r, p_2^r)}{\ell}.$$

By definition of the Δ -symbol, we write $E^n = E_p^n + \frac{8\pi i^\kappa}{\log^2(q^r)} E_c^n$ with

$$E_p^n := \frac{4}{\log^2(q^r)} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \frac{\log^2 p}{p} (\widehat{\Phi}_1 \widehat{\Phi}_2) \left(\frac{\log p}{\log(q^r)} \right)$$

and

$$E_c^n := \sum_{\substack{c \geq 1 \\ q \mid c}} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid q}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid q}} \frac{\log p_1}{\sqrt{p_1}} \frac{\log p_2}{\sqrt{p_2}} \widehat{\Phi}_1 \left(\frac{\log p_1}{\log(q^r)} \right) \widehat{\Phi}_2 \left(\frac{\log p_2}{\log(q^r)} \right) \times \frac{S(p_1^r, p_2^r; c)}{c} J_{\kappa-1} \left(\frac{4\pi \sqrt{p_1^r p_2^r}}{c} \right).$$

We remove the condition $p \nmid q$ from E_p^n at an admissible cost and obtain, after integration by parts,

$$E_p^n = 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) du + O \left(\frac{1}{\log^2(q^r)} \right). \tag{5.2}$$

Using Corollary 3.5, we get

$$E_c^n \ll \frac{1}{\log^2(q^r)} \tag{5.3}$$

when $\nu \leq 1/r^2$. Finally, using Lemma 3.13, we see that E° is an admissible error term for $\nu < 1/r$ so that equations (5.2) and (5.3) lead to

$$\mathbb{C}_q^h = 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) \, du + O\left(\frac{1}{\log^2(q^r)}\right). \tag{5.4}$$

Let $\{i, j\} = \{1, 2\}$. We prove next that each $\mathbb{E}_q^h(P_q^1[\Phi_i; r] P_q^2[\Phi_j; r, m])$ is an error term when $\nu < 1/r^2$. Using Proposition 2.2 and Lemma 3.13 we have

$$\begin{aligned} & \mathbb{E}_q^h(P_q^1[\Phi_i; r] P_q^2[\Phi_j; r, m]) \\ &= \frac{8\pi i^\kappa}{\log^2(q^r)} \sum_{\substack{c \geq 1 \\ q|c}} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid c}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid c}} \frac{\log p_1}{\sqrt{p_1}} \frac{\log p_2}{p_2} \widehat{\Phi}_i\left(\frac{\log p_1}{\log(q^r)}\right) \widehat{\Phi}_j\left(\frac{\log p_2}{\log(q^{r/2})}\right) \\ & \quad \times \frac{S(p_1^r, p_2^{2r-2m}; c)}{c} J_{\kappa-1}\left(\frac{4\pi \sqrt{p_1^r p_2^{2r-2m}}}{c}\right) + O\left(\frac{1}{\log(q^r)}\right)^2. \end{aligned}$$

We use Corollary 3.5 to conclude that

$$\mathbb{E}_q^h(P_q^1[\Phi_i; r] P_q^2[\Phi_j; r, m]) \ll \frac{1}{\log q} \tag{5.5}$$

when $\nu < 1/r^2$. Finally, $\mathbb{E}_q^h(P_q^2[\Phi_1; r, m_1] P_q^2[\Phi_2; r, m_2])$ is shown to be an error term in the same way. \square

Using Lemmas 5.3 and 5.4, Theorem 4.1, Hypothesis Nice(r, f) and Remark 3.12, we prove the following theorem.

Theorem 5.7. *Let $r \geq 1$. Let Φ_1 and Φ_2 in $\mathcal{S}_\nu(\mathbb{R})$. We assume that Hypothesis Nice(r, f) holds for any prime number q and any $f \in H_\kappa^*(q)$ and also that θ is admissible. If $\nu < \nu_{2, \max}(r, \kappa, \theta)$, then*

$$\begin{aligned} \mathbb{E}_\infty^h(D_2[\Phi_1, \Phi_2; r]) &= \left[\widehat{\Phi}_1(0) + \frac{(-1)^{r+1}}{2} \Phi_1(0) \right] \left[\widehat{\Phi}_2(0) + \frac{(-1)^{r+1}}{2} \Phi_2(0) \right] \\ & \quad + 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) \, du - 2 \widehat{\Phi}_1 \widehat{\Phi}_2(0) \\ & \quad + \left((-1)^r + \frac{\mathbb{1}_{2\mathbb{N}+1}(r)}{2} \right) \Phi_1(0) \Phi_2(0). \end{aligned}$$

Some comments are given in Remark 1.6 on page 976.

5.2 Signed asymptotic expectation of the two-level density and signed asymptotic variance

In this part, r is odd.

Lemma 5.8. *Let $\Phi_1, \Phi_2 \in \mathcal{S}_\nu(\mathbb{R})$ and $r \geq 1$ be an odd integer. If $\nu < 1/(2r^2)$, then*

$$\mathbb{E}_\infty^{\text{h},\varepsilon} (D_1[\Phi_1; r]D_1[\Phi_2; r]) = E[\Phi_1; r]E[\Phi_2; r] + 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) du.$$

Remark 5.9. By Theorem 4.4 and Lemma 5.8 we have

$$\begin{aligned} &\mathbb{E}_\infty^{\text{h},\varepsilon} (D_1[\Phi_1; r]D_1[\Phi_2; r]) - E[\Phi_1; r]E[\Phi_2; r] \\ &= \mathbb{E}_\infty^{\text{h},\varepsilon} (D_1[\Phi_1; r]D_1[\Phi_2; r]) - \mathbb{E}_\infty^{\text{h},\varepsilon} (D_1[\Phi_1; r]) \mathbb{E}_\infty^{\text{h},\varepsilon} (D_1[\Phi_2; r]). \end{aligned}$$

Thus,

$$\mathbb{C}_\infty^{\text{h},\varepsilon} (D_1[\Phi_1; r], D_1[\Phi_2; r]) := 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) du$$

is the signed asymptotic covariance of $D_1[\Phi_1; r]$ and $D_1[\Phi_2; r]$. In particular, taking $\Phi_1 = \Phi_2$, we obtain the signed asymptotic variance.

Theorem 5.10. *Let $\Phi \in \mathcal{S}_\nu(\mathbb{R})$ and $r \geq 1$ be an odd integer. If $\nu < 1/(2r^2)$, then the signed asymptotic variance of $D_1[\Phi; r]$ is*

$$\mathbb{V}_\infty^{\text{h},\varepsilon} (D_1[\Phi; r]) = 2 \int_{\mathbb{R}} |u| \widehat{\Phi}^2(u) du.$$

Proof of Lemma 5.8. From Proposition 3.8 and (3.20), we obtain

$$\begin{aligned} &\mathbb{E}_q^{\text{h},\varepsilon} (D_{1,q}[\Phi_1; r]D_{1,q}[\Phi_2; r]) \\ &= E[\Phi_1; r]E[\Phi_2; r] + \mathbb{C}_q^{\text{h},\varepsilon} \\ &\quad + \sum_{\substack{(i,j) \in \{1,2\}^2 \\ i \neq j}} \sum_{m=0}^{r-1} (-1)^m \mathbb{E}_q^{\text{h},\varepsilon} (P_q^1[\Phi_i; r]P_q^2[\Phi_j; r, m]) \\ &\quad + \sum_{m_1=0}^{r-1} \sum_{m_2=0}^{r-1} (-1)^{m_1+m_2} \mathbb{E}_q^{\text{h},\varepsilon} (P_q^2[\Phi_1; r, m_1]P_q^2[\Phi_2; r, m_2]) \\ &\quad + O\left(\frac{1}{\log(q^r)}\right) \end{aligned} \tag{5.6}$$

with

$$\mathbb{C}_q^{h,\varepsilon} := \mathbb{E}_q^{h,\varepsilon} (P_q^1[\Phi_1; r]P_q^1[\Phi_2; r]).$$

Assume that $\nu < 1/r^2$. Equations (2.18), (5.4) and Proposition 2.2 lead to

$$\mathbb{C}_q^{h,\varepsilon} = 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) du - \varepsilon \times \varepsilon(\kappa, r)(G^n - 4G^o) \tag{5.7}$$

with

$$G^n := \frac{4\sqrt{q}}{\log^2(q^r)} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid q}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid q}} \frac{\log p_1}{\sqrt{p_1}} \frac{\log p_2}{\sqrt{p_2}} \widehat{\Phi}_1\left(\frac{\log p_1}{\log(q^r)}\right) \times \widehat{\Phi}_2\left(\frac{\log p_2}{\log(q^r)}\right) \Delta_q(p_1^r q, p_2^r)$$

and

$$G^o := \frac{1}{\sqrt{q} \log^2(q^r)} \sum_{\substack{p_1 \in \mathcal{P} \\ p_1 \nmid q}} \sum_{\substack{p_2 \in \mathcal{P} \\ p_2 \nmid q}} \frac{\log p_1}{\sqrt{p_1}} \frac{\log p_2}{\sqrt{p_2}} \widehat{\Phi}_1\left(\frac{\log p_1}{\log(q^r)}\right) \times \widehat{\Phi}_2\left(\frac{\log p_2}{\log(q^r)}\right) \sum_{\ell|q^\infty} \frac{\Delta_q(\ell^2 p_1^r, p_2^r q)}{\ell}.$$

Lemma 3.10 implies that if $\nu < 1/(2r^2)$, then

$$G^n \ll \frac{q^{\nu r[r(\kappa-1)+1]/2}}{q^{(\kappa-1)/2}}, \tag{5.8}$$

hence G^n is an error term when $\nu \leq 1/(2r^2)$. Lemma 3.13 implies

$$G^o \ll q^{-3/2+\nu r+\varepsilon}, \tag{5.9}$$

which is an error term. The estimates (5.8), (5.9) and the equation (5.7) imply that

$$\mathbb{C}_\infty^{h,\varepsilon} = 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) du \tag{5.10}$$

for $\nu \leq 1/(2r(r+2))$. Next, we prove that each $\mathbb{E}_q^{h,\varepsilon} (P_q^1[\Phi_i; r]P_q^2[\Phi_j; r, m])$ is an error term when $\nu \leq 1/(2r^2)$. From equations (2.18) and (5.5), we obtain

$$\begin{aligned} & \mathbb{E}_q^{h,\varepsilon} (P_q^1[\Phi_i; r]P_q^2[\Phi_j; r, m]) \\ &= -\varepsilon \times \varepsilon(\kappa, r) \sqrt{q} \sum_{f \in H_\kappa^*(q)} \lambda_f(q) P_q^1[\Phi_i; r]P_q^2[\Phi_j; r, m] + O\left(\frac{1}{\log q}\right). \end{aligned} \tag{5.11}$$

We use Proposition 2.2 and Lemmas 3.13 and 3.10 to get

$$\begin{aligned} \sqrt{q} \sum_{f \in H_{\kappa}^*(q)}^h \lambda_f(q) P_q^1[\Phi_i; r] P_q^2[\Phi_j; r, m] \\ \ll \frac{q^{\nu r(2r-m+2)/4-1/4}}{\log^2 q} + \frac{q^{(\nu r-1)/2+\varepsilon}}{\log q}. \end{aligned} \tag{5.12}$$

It follows from (5.12) and (5.11) that

$$\mathbb{E}_{\infty}^{h,\varepsilon} (P_q^1[\Phi_i; r] P_q^2[\Phi_j; r, m]) = 0 \tag{5.13}$$

for $\nu \leq 1/(2r(r+1))$. In the same way, we have, for ν in the previous range,

$$\mathbb{E}_{\infty}^{h,\varepsilon} (P_q^2[\Phi_1; r, m_1] P_q^2[\Phi_2; r, m_2]) = 0. \tag{5.14}$$

According to the estimates (5.10), (5.13), (5.14), the equation (5.6) entails the desired result. \square

Using Lemmas 5.3 and 5.8, Theorem 4.4, Hypothesis Nice(r, f) and (3.20), we prove the following theorem.

Theorem 5.11. *Let $f \in H_{\kappa}^*(q)$, Φ_1, Φ_2 be in $\mathcal{S}_{\nu}(\mathbb{R})$ and $r \geq 1$ be an odd integer. If $\nu < 1/(2r(r+1))$, then*

$$\begin{aligned} \mathbb{E}_{\infty}^{h,\varepsilon} (D_2[\Phi_1, \Phi_2; r]) &= \left[\widehat{\Phi}_1(0) + \frac{1}{2} \Phi_1(0) \right] \left[\widehat{\Phi}_2(0) + \frac{1}{2} \Phi_2(0) \right] \\ &\quad + 2 \int_{\mathbb{R}} |u| \widehat{\Phi}_1(u) \widehat{\Phi}_2(u) \, du - 2 \widehat{\Phi}_1 \widehat{\Phi}_2(0) \\ &\quad - \Phi_1(0) \Phi_2(0) + \mathbb{1}_{\{-1\}}(\varepsilon) \Phi_1(0) \Phi_2(0). \end{aligned}$$

Remark 5.12. Remark 4.3 together with Theorem 5.11 and a result of Katz & Sarnak (see [21, Theorem A.D.2.2] or [28, Theorem 3.2]) imply that the symmetry type of $\mathcal{F}_r^{\varepsilon}$ is as in Table 2.

$\varepsilon \backslash r$	even	odd
-1		SO(odd)
1	Sp	SO(even)

Table 2. Symmetry type of $\mathcal{F}_r^{\varepsilon}$.

6 First asymptotic moments of the one-level density

In this section, we compute the asymptotic m -th moment of the one level density, namely

$$\mathbb{M}_{\infty,m}^h(D_{1,q}[\Phi; r]) := \lim_{\substack{q \in \mathcal{P} \\ q \rightarrow +\infty}} \mathbb{M}_{q,m}^h(D_{1,q}[\Phi; r])$$

where

$$\mathbb{M}_{q,m}^h(D_{1,q}[\Phi; r]) = \mathbb{E}_q^h \left((D_{1,q}[\Phi; r] - \mathbb{E}_q^h(D_{1,q}[\Phi; r]))^m \right)$$

for m small enough (regarding to the size of the support of Φ). The end of this section is devoted to the proof of Theorem E. Note that we can assume that $m \geq 3$ since the work has already been done for $m = 1$ and $m = 2$. Thanks to equation (4.14) and Proposition 3.8, we have

$$\mathbb{M}_{q,m}^h(D_{1,q}[\Phi; r]) = \mathbb{E}_q^h \left((P_q^1[\Phi; r] + P_q^2[\Phi; r] + R_q) \right)$$

where R_q is a function on $H_\kappa^*(q)$ satisfying

$$R_q(f) = O_f \left(\frac{1}{q} \right)$$

for any f in $H_\kappa^*(q)$ and

$$\begin{aligned} P_q^2[\Phi; r](f) &:= -\frac{2}{\log(q^r)} \sum_{j=0}^{r-1} (-1)^j \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \lambda_f(p^{2(r-j)}) \frac{\log p}{p} \widehat{\Phi} \left(\frac{2 \log p}{\log(q^r)} \right) \\ &= -\frac{2}{\log(q^r)} \sum_{j=1}^r (-1)^{r-j} \sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} \lambda_f(p^{2j}) \frac{\log p}{p} \widehat{\Phi} \left(\frac{2 \log p}{\log(q^r)} \right). \end{aligned} \tag{6.1}$$

We follow [15, Appendix B] in order to remove properly the error term. It gives

$$\mathbb{M}_{q,m}^h(D_{1,q}[\Phi; r]) = \sum_{\ell=0}^m \binom{m}{\ell} \mathbb{E}_q^h \left(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell \right) + O \left(\frac{1}{q} \right).$$

Thus, an asymptotic formula for $\mathbb{M}_{q,m}^h(D_{1,q}[\Phi; r])$ directly follows from the next proposition.

Proposition 6.1. *Let $r \geq 1$ be any integer. We assume that Hypothesis Nice(r, f) holds for any prime number q and any primitive holomorphic cusp form of level q and even weight κ . Let $\alpha \geq 0$ and $\ell \geq 0$ be any integers.*

- If $\ell \geq 1$ and $\ell v < 4/r^2$, then

$$\mathbb{E}_q^h \left(P_q^2[\Phi; r]^\ell \right) = O \left(\frac{1}{\log q} \right).$$

- If $1 \leq \ell \leq m - 1$ and $mv < 4/(r(r + 2))$, then

$$\mathbb{E}_q^h \left(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell \right) = O \left(\frac{1}{\log q} \right).$$

- If $\ell \geq 1$ and $\ell v < 4/(r(r + 2))$, then

$$\begin{aligned} & \mathbb{E}_q^h \left(P_q^1[\Phi; r]^\ell \right) \\ &= \begin{cases} O \left(\frac{1}{\log^2(q)} \right), & \text{if } \alpha \text{ is odd,} \\ \left(2 \int_{\mathbb{R}} |u| \widehat{\Phi}^2(u) du \right)^{\ell/2} \times \frac{\ell!}{2^{\ell/2} \left(\frac{\ell}{2}\right)!} + O \left(\frac{1}{\log^2(q)} \right), & \text{otherwise.} \end{cases} \end{aligned}$$

6.1 On a useful combinatorial identity

In order to use the multiplicative properties of Hecke eigenvalues in the proof of Proposition 6.1, we want to reorder some sums over many primes to sums over distinct primes. We follow the work of Hughes & Rudnick [16, §7] (see also [14] and the work of Soshnikov [30]) to achieve this. Let $P(\alpha, s)$ be the set of surjective functions

$$\sigma: \{1, \dots, \alpha\} \twoheadrightarrow \{1, \dots, s\}$$

such that for any $j \in \{1, \dots, \alpha\}$, either $\sigma(j) = 1$ or there exists $k < j$ such that $\sigma(j) = \sigma(k) + 1$. This can be viewed as the number of partitions of a set of α elements into s nonempty subsets. By definition, the cardinality of $P(\alpha, s)$ is the Stirling number of second kind [31, §1.4]. For any $j \in \{1, \dots, s\}$, let

$$\varpi_j^{(\sigma)} := \#(\sigma^{-1}(\{j\})).$$

Note that

$$\varpi_j^{(\sigma)} \geq 1 \quad \text{for any } 1 \leq j \leq s \quad \text{and} \quad \sum_{j=1}^s \varpi_j^{(\sigma)} = \alpha. \quad (6.2)$$

The following lemma is Lemma 7.3 of [16, §7].

Lemma 6.2. *If g is any function of m variables, then*

$$\sum_{j_1, \dots, j_m} g(x_{j_1}, \dots, x_{j_m}) = \sum_{s=1}^m \sum_{\sigma \in P(m,s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} g(x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(m)}}).$$

6.2 Proof of the first bullet of Proposition 6.1

By Definition (6.1), we have

$$\begin{aligned} \mathbb{E}_q^h \left(P_q^2[\Phi; r]^\ell \right) &= \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{1 \leq j_1, \dots, j_\ell \leq r} (-1)^{\ell r - (j_1 + \dots + j_\ell)} \\ &\quad \times \sum_{\substack{p_1, \dots, p_\ell \in \mathcal{P} \\ q \nmid p_1 \dots p_\ell}} \left(\prod_{i=1}^{\ell} \frac{\log p_i}{p_i} \widehat{\Phi} \left(\frac{2 \log p_i}{\log(q^r)} \right) \right) \\ &\quad \times \mathbb{E}_q^h \left(\prod_{i=1}^{\ell} \lambda_f(p_i^{2j_i}) \right). \end{aligned} \quad (6.3)$$

Writing $\{\widehat{p}_i\}_{i \geq 1}$ for the increasing sequence of prime numbers except q , we have

$$\begin{aligned} &\sum_{\substack{p_1, \dots, p_\ell \in \mathcal{P} \\ q \nmid p_1 \dots p_\ell}} \left(\prod_{i=1}^{\ell} \frac{\log p_i}{p_i} \widehat{\Phi} \left(\frac{2 \log p_i}{\log(q^r)} \right) \right) \mathbb{E}_q^h \left(\prod_{i=1}^{\ell} \lambda_f(p_i^{2j_i}) \right) \\ &= \sum_{i_1, \dots, i_\ell} \left(\prod_{\ell=1}^{\ell} \frac{\log \widehat{p}_{i_\ell}}{\widehat{p}_{i_\ell}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{i_\ell}}{\log(q^r)} \right) \right) \mathbb{E}_q^h \left(\prod_{\ell=1}^{\ell} \lambda_f(\widehat{p}_{i_\ell}^{2j_\ell}) \right). \end{aligned} \quad (6.4)$$

Using Lemma 6.2, we rewrite the right sum in (6.4) as

$$\begin{aligned} &\sum_{s=1}^{\ell} \sum_{\sigma \in P(\ell, s)} \sum_{\substack{k_1, \dots, k_s \\ \text{distinct}}} \left(\prod_{i=1}^{\ell} \frac{\log \widehat{p}_{k_{\sigma(i)}}}{\widehat{p}_{k_{\sigma(i)}}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{k_{\sigma(i)}}}{\log(q^r)} \right) \right) \mathbb{E}_q^h \left(\prod_{i=1}^{\ell} \lambda_f(\widehat{p}_{k_{\sigma(i)}}^{2j_i}) \right) \\ &= \sum_{s=1}^{\ell} \sum_{\sigma \in P(\ell, s)} \sum_{\substack{k_1, \dots, k_s \\ \text{distinct}}} \left(\prod_{u=1}^s \left(\frac{\log \widehat{p}_{k_u}}{\widehat{p}_{k_u}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{k_u}}{\log(q^r)} \right) \right)^{\varpi_u^{(\sigma)}} \right) \\ &\quad \times \mathbb{E}_q^h \left(\prod_{\substack{1 \leq u \leq s \\ 1 \leq j \leq r}} \lambda_f(\widehat{p}_{k_u}^{2j})^{\varpi_{u,j}^{(\sigma)}} \right) \end{aligned} \quad (6.5)$$

where

$$\varpi_{u,j}^{(\sigma)} := \#\{1 \leq i \leq \ell : \sigma(i) = u, j_i = j\}$$

for any $1 \leq u \leq s$ and any $1 \leq j \leq r$. Now, we show that

$$\sum_{s=1}^{\ell-1} \sum_{\sigma \in P(\ell,s)} \sum_{\substack{k_1, \dots, k_s \\ \text{distinct}}} \left(\prod_{u=1}^s \left(\frac{\log \widehat{p}_{k_u}}{\widehat{p}_{k_u}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{k_u}}{\log(q^r)} \right) \right)^{\varpi_u^{(\sigma)}} \right) \times \mathbb{E}_q^h \left(\prod_{\substack{1 \leq u \leq s \\ 1 \leq j \leq r}} \lambda_f(\widehat{p}_{k_u}^{2j})^{\varpi_{u,j}^{(\sigma)}} \right) \ll \log^{\ell-1}(q). \tag{6.6}$$

For $s < \ell$ and $\sigma \in P(\ell, s)$, we use (2.3) together with (3.19) to obtain that the left-hand side of the previous equation is bounded by

$$\sum_{s=1}^{\ell-1} \sum_{\sigma \in P(\ell,s)} \sum_{\substack{k_1, \dots, k_s \\ \text{distinct}}} \prod_{u=1}^s \left(\frac{\log \widehat{p}_{k_u}}{\widehat{p}_{k_u}} \left| \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{k_u}}{\log(q^r)} \right) \right| \right)^{\varpi_u^{(\sigma)}}. \tag{6.7}$$

Since $s < \ell$, equation (6.2) implies that $\varpi_u^{(\sigma)} > 1$ for some $1 \leq u \leq s$. These values lead to convergent, hence bounded, sums. Let

$$d^{(\sigma)} := \# \left\{ 1 \leq u \leq s : \varpi_u^{(\sigma)} = 1 \right\} \in \{0, \dots, \ell - 1\},$$

then

$$\begin{aligned} & \sum_{s=1}^{\ell-1} \sum_{\sigma \in P(\ell,s)} \sum_{\substack{k_1, \dots, k_s \\ \text{distinct}}} \prod_{u=1}^s \left(\frac{\log \widehat{p}_{k_u}}{\widehat{p}_{k_u}} \left| \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{k_u}}{\log(q^r)} \right) \right| \right)^{\varpi_u^{(\sigma)}} \\ & \ll \sum_{s=1}^{\ell-1} \sum_{\sigma \in P(\ell,s)} \sum_{\substack{k_1, \dots, k_d \\ \text{distinct}}} \prod_{u=1}^{d^{(\sigma)}} \left(\frac{\log \widehat{p}_{k_u}}{\widehat{p}_{k_u}} \left| \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{k_u}}{\log(q^r)} \right) \right| \right) \\ & \ll \log^{\ell-1}(q). \end{aligned}$$

We have altogether

$$\begin{aligned} \mathbb{E}_q^h \left(P_q^2[\Phi; r]^\ell \right) &= \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{1 \leq j_1, \dots, j_\ell \leq r} (-1)^{\ell r - (j_1 + \dots + j_\ell)} \\ & \times \sum_{\substack{k_1, \dots, k_\ell \\ \text{distinct}}} \left(\prod_{u=1}^{\ell} \left(\frac{\log \widehat{p}_{k_u}}{\widehat{p}_{k_u}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{k_u}}{\log(q^r)} \right) \right) \right) \\ & \times \mathbb{E}_q^h \left(\lambda_f \left(\prod_{u=1}^{\ell} \widehat{p}_{k_u}^{2j_u} \right) \right) + O \left(\frac{1}{\log q} \right), \tag{6.8} \end{aligned}$$

since the only element of $P(\ell, \ell)$ is the identity function. By Lemmas 3.9 and 3.10, we have

$$\mathbb{E}_q^h \left(\lambda_f \left(\prod_{u=1}^{\ell} \widehat{p}_{k_u}^{2j_u} \right) \right) \ll \frac{1}{q} \prod_{u=1}^{\ell} \widehat{p}_{k_u}^{j_u/2} \log \widehat{p}_{k_u},$$

hence the first term in the right-hand side of (6.8) is bounded by a negative power of q when $\ell \nu r^2 < 4$.

6.3 Proof of the third bullet of Proposition 6.1

By Proposition 3.8, we have

$$\begin{aligned} \mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) &= \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{\substack{p_1, \dots, p_\ell \in \mathcal{P} \\ p_1, \dots, p_\ell \nmid q}} \left(\prod_{i=1}^{\ell} \frac{\log p_i}{\sqrt{p_i}} \widehat{\Phi} \left(\frac{\log p_i}{\log q^r} \right) \right) \\ &\quad \times \mathbb{E}_q^h \left(\prod_{i=1}^{\ell} \lambda_f(p_i^r) \right). \end{aligned} \tag{6.9}$$

Using Lemma 6.2, we rewrite equation (6.9) as

$$\begin{aligned} &\mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) \\ &= \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{s=1}^{\ell} \sum_{\sigma \in P(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \left(\prod_{j=1}^{\ell} \left(\frac{\log \widehat{p}_{i_{\sigma(j)}}}{\sqrt{\widehat{p}_{i_{\sigma(j)}}}} \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_{\sigma(j)}}}{\log(q^r)} \right) \right) \right) \\ &\quad \times \mathbb{E}_q^h \left(\prod_{j=1}^{\ell} \lambda_f(\widehat{p}_{i_{\sigma(j)}}^r) \right) \\ &= \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{s=1}^{\ell} \sum_{\sigma \in P(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \left(\prod_{u=1}^s \left(\frac{\log \widehat{p}_{i_u}}{\sqrt{\widehat{p}_{i_u}}} \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log q^r} \right) \right)^{\varpi_u^{(\sigma)}} \right) \\ &\quad \times \mathbb{E}_q^h \left(\prod_{u=1}^s \lambda_f(\widehat{p}_{i_u}^r)^{\varpi_u^{(\sigma)}} \right). \end{aligned} \tag{6.10}$$

It follows from (2.11) and (2.12) that

$$\lambda_f(\widehat{p}_{i_u}^r)^{\varpi_u^{(\sigma)}} = \sum_{j_u=0}^{r \varpi_u^{(\sigma)}} x(\varpi_u^{(\sigma)}, r, j_u) \lambda_f(\widehat{p}_{i_u}^{j_u}).$$

Since $u \neq v$ implies that $\widehat{p}_{i_u} \neq \widehat{p}_{i_v}$, equation (6.10) becomes

$$\begin{aligned} & \mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) \\ &= \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{s=1}^\ell \sum_{\sigma \in P(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \left(\prod_{u=1}^s \left(\frac{\log \widehat{p}_{i_u}}{\sqrt{\widehat{p}_{i_u}}} \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right)^{\varpi_u^{(\sigma)}} \right) \\ & \quad \times \sum_{\substack{j_1, \dots, j_s \\ 0 \leq j_u \leq r \varpi_u^{(\sigma)}}} \left(\prod_{u=1}^s x(\varpi_u^{(\sigma)}, r, j_u) \right) \mathbb{E}_q^h \left(\lambda_f \left(\prod_{u=1}^s \widehat{p}_{i_u}^{j_u} \right) \right). \end{aligned} \tag{6.11}$$

Using Proposition 2.2 and Lemmas 3.10 and 3.9, we get

$$\mathbb{E}_q^h \left(\lambda_f \left(\prod_{u=1}^s \widehat{p}_{i_u}^{j_u} \right) \right) = \prod_{u=1}^s \delta_{j_u, 0} + O \left(\frac{1}{q} \prod_{u=1}^s \widehat{p}_{i_u}^{j_u/4} \log \widehat{p}_{i_u} \right),$$

hence

$$\mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) = \text{TP} + O(\text{TE}) \tag{6.12}$$

with

$$\text{TP} := \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{s=1}^\ell \sum_{\sigma \in P(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \left(\frac{\log \widehat{p}_{i_u}}{\sqrt{\widehat{p}_{i_u}}} \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right)^{\varpi_u^{(\sigma)}} x(\varpi_u^{(\sigma)}, r, 0) \tag{6.13}$$

and

$$\text{TE} := \frac{1}{q \log^\ell(q^r)} \sum_{s=1}^\ell \sum_{\sigma \in P(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \left(\widehat{p}_{i_u}^{(r-2)/4} \log^2 \widehat{p}_{i_u} \left| \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right| \right)^{\varpi_u^{(\sigma)}}. \tag{6.14}$$

We have

$$\text{TE} = \frac{1}{q \log^\ell(q^r)} \left(\sum_{\substack{p \in \mathcal{P} \\ p \nmid q}} p^{(r-2)/4} \log^2 p \left| \widehat{\Phi} \left(\frac{\log p}{\log(q^r)} \right) \right| \right)^\ell \ll q^{\ell r \nu(r+2)/4 - 1} \tag{6.15}$$

so that TE is an error term when

$$\ell r \nu(r + 2) < 4. \tag{6.16}$$

We assume from now on that this condition is satisfied. According to (2.14) (recall that $r \geq 1$), we rewrite (6.13) as

$$\text{TP} = \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{s=1}^\ell \sum_{\sigma \in P^{\geq 2}(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \left(\frac{\log \widehat{p}_{i_u}}{\sqrt{\widehat{p}_{i_u}}} \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right)^{\varpi_u^{(\sigma)}} x(\varpi_u^{(\sigma)}, r, 0) \tag{6.17}$$

where

$$P^{\geq 2}(\ell, s) := \left\{ \sigma \in P(\ell, s) : \forall u \in \{1, \dots, s\}, \varpi_u^{(\sigma)} \geq 2 \right\}.$$

Moreover, if for at least one σ and at least one u (say u_0) we have $\varpi_{u_0}^{(\sigma)} \geq 3$, then

$$\begin{aligned}
 & \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \left(\frac{\log \widehat{p}_{i_u}}{\sqrt{\widehat{p}_{i_u}}} \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right)^{\varpi_u^{(\sigma)}} x(\varpi_u^{(\sigma)}, r, 0) \\
 & \ll \left(\sum_{\substack{p \in \mathcal{P} \\ p \leq q^{rv}}} \frac{\log^3(p)}{p^{3/2}} \right) \prod_{\substack{u=1 \\ u \neq u_0}}^s \left(\sum_{\substack{p_u \in \mathcal{P} \\ p_u \leq q^{rv}}} \frac{\log^2(p_u)}{p_u} \right) \\
 & \ll (\log q)^{2s-2}. \tag{6.18}
 \end{aligned}$$

But, from (6.2) and the fact that each $\varpi_j^{(\sigma)} \geq 2$, we deduce

$$2s \leq \sum_{j=1}^s \varpi_j^{(\sigma)} = \ell,$$

hence $(\log q)^{2s-2} \ll (\log q)^{\ell-2}$. This fact and the estimate (6.18) combined with (6.17) imply that

$$\begin{aligned}
 \text{TP} &= \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{s=1}^\ell \sum_{\sigma \in P^2(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \left(\frac{\log \widehat{p}_{i_u}}{\sqrt{\widehat{p}_{i_u}}} \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log q^r} \right) \right)^{\varpi_u^{(\sigma)}} x(\varpi_u^{(\sigma)}, r, 0) \\
 &+ O\left(\frac{1}{\log^2(q)}\right) \tag{6.19}
 \end{aligned}$$

where

$$P^2(\ell, s) := \left\{ \sigma \in P(\ell, s) : \forall u \in \{1, \dots, s\}, \varpi_u^{(\sigma)} = 2 \right\}.$$

From (6.19), (6.15) and (6.12), we deduce

$$\mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) = \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{s=1}^\ell \sum_{\sigma \in P^2(\ell, s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \frac{\log^2(\widehat{p}_{i_u})}{\widehat{p}_{i_u}} \widehat{\Phi}^2\left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)}\right) + O\left(\frac{1}{\log^2(q)}\right)$$

since $x(2, r, 0) = 1$ according to (2.14). Note in particular that according to (6.2) the previous sum is zero if ℓ is odd. Thus we can assume now that ℓ is even and get

$$\mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) = \frac{(-2)^\ell}{\log^\ell(q^r)} \sum_{\sigma \in P^2(\ell, \ell/2)} \sum_{\substack{i_1, \dots, i_{\ell/2} \\ \text{distinct}}} \prod_{u=1}^{\ell/2} \frac{\log^2(\widehat{p}_{i_u})}{\widehat{p}_{i_u}} \widehat{\Phi}^2\left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)}\right) + O\left(\frac{1}{\log^2(q)}\right). \tag{6.20}$$

However, summing over all the possible $(i_1, \dots, i_{\ell/2})$ instead of the one with distinct indices reintroduces convergent sums that enter the error term because of the $1/\log^\ell(q^r)$ factor. It follows that (6.20) becomes

$$\mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) = \left[\frac{4}{\log^2(q^r)} \sum_{p \in \mathcal{P}} \frac{\log^2(p)}{p} \widehat{\Phi}^2\left(\frac{\log p}{\log(q^r)}\right) \right]^{\ell/2} \#P^2(\ell, \ell/2) + O\left(\frac{1}{\log^2(q)}\right). \tag{6.21}$$

Taking $\ell = 2$ (we already proved that the second moment is finite, see Section 5.1), we get

$$\mathbb{E}_q^h(P_q^1[\Phi; r]^\ell) = \mathbb{E}_q^h(P_q^1[\Phi; r]^2)^{\ell/2} \#P^2(\ell, \ell/2) + O\left(\frac{1}{\log^2(q)}\right)$$

according to (6.21). We conclude by noticing

$$\#P^2(\ell, \ell/2) = \frac{\ell!}{2^{\ell/2} \left(\frac{\ell}{2}\right)!}$$

(see [32, Example 5.2.6 and Exercise 5.43]).

6.4 Proof of the second bullet of Proposition 6.1

We mix the two techniques, which have been used to prove the first and third bullets of Proposition 6.1. We get following the same lines and thanks to Lemma 6.2

$$\begin{aligned} & \mathbb{E}_q^h(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell) \\ &= \frac{(-2)^m}{\log^m(q^r)} \sum_{1 \leq j_1, \dots, j_\ell \leq r} (-1)^{\ell r - (j_1 + \dots + j_\ell)} \\ & \quad \times \sum_{s=1}^m \sum_{\sigma \in P(m,s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \left(\frac{\log \widehat{w}_u^{(\sigma,1)} + \widehat{w}_u^{(\sigma,2)}(\widehat{p}_{i_u})}{\widehat{p}_{i_u}^{\widehat{w}_u^{(\sigma,1)}/2 + \widehat{w}_u^{(\sigma,2)}}} \right) \\ & \quad \times \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right)^{\widehat{w}_u^{(\sigma,1)}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{i_u}}{\log(q^r)} \right)^{\widehat{w}_u^{(\sigma,2)}} \\ & \quad \times \mathbb{E}_q^h \left(\prod_{u=1}^s \left(\lambda_f(\widehat{p}_{i_u}^r)^{\widehat{w}_u^{(\sigma,1)}} \prod_{j=1}^r \lambda_f(\widehat{p}_{i_u}^{2j})^{\widehat{w}_{u,j}^{(\sigma,2)}} \right) \right) \end{aligned}$$

where

$$\begin{aligned} \widehat{w}_u^{(\sigma,1)} &:= \#\{i \in \{1, \dots, m - \ell\} : \sigma(i) = u\}, \\ \widehat{w}_u^{(\sigma,2)} &:= \#\{i \in \{1, \dots, \alpha\} : \sigma(m - \ell + i) = u\}, \\ \widehat{w}_{u,j}^{(\sigma,2)} &:= \#\{i \in \{1, \dots, \alpha\} : \sigma(m - \ell + i) = u \text{ and } j_i = j\} \end{aligned}$$

for any $1 \leq u \leq s$, any $1 \leq j \leq r$ and any $\sigma \in P(m, s)$. Note that these numbers satisfy

$$\sum_{u=1}^s \left(\widehat{w}_u^{(\sigma,1)} + \widehat{w}_u^{(\sigma,2)} \right) = m \tag{6.22}$$

and

$$\sum_{j=1}^r \widehat{w}_{u,j}^{(\sigma,2)} = \widehat{w}_u^{(\sigma,2)} \tag{6.23}$$

for any $1 \leq u \leq r$ and any $\sigma \in P(m, s)$ by definition. They also satisfy

$$\forall \sigma \in P(m, s), \forall u \in \{1, \dots, s\}, \quad \widehat{w}_u^{(\sigma,1)} + \widehat{w}_u^{(\sigma,2)} \geq 1 \tag{6.24}$$

since any $\sigma \in P(m, s)$ is surjective and

$$\forall \sigma \in P(m, s), \forall i \in \{1, 2\}, \exists u_{i,\sigma} \in \{1, \dots, s\}, \quad \varpi_{u_{i,\sigma}}^{(\sigma,i)} \geq 1 \quad (6.25)$$

since $\ell \geq 1$ and $m - \ell \geq 1$. The strategy is to estimate individually each term of the σ -sum. Thus, we fix some integers j_1, \dots, j_ℓ in $\{1, \dots, r\}$, some integer s in $\{1, \dots, r\}$ and some map σ in $P(m, s)$.

First case. $\forall u \in \{1, \dots, s\}, \varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)} \leq 1$.

Let us remark that if $\varpi_u^{(\sigma,2)} = 1$ for some $1 \leq u \leq s$, then there exists a unique $1 \leq j_{i_u} \leq r$ depending on σ such that $\varpi_{u,j_{i_u}}^{(\sigma,2)} = 1$ and $\varpi_{u,j}^{(\sigma,2)} = 0$ for any $1 \leq j \neq j_{i_u} \leq r$ according to (6.23). Thus,

$$\begin{aligned} & \prod_{u=1}^s \left(\lambda_f(\widehat{p}_{i_u}^r) \varpi_u^{(\sigma,1)} \prod_{j=1}^r \left(\lambda_f(\widehat{p}_{i_u}^{2j}) \varpi_{u,j}^{(\sigma,2)} \right) \right) \\ &= \lambda_f \left(\prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (2,0)}} \widehat{p}_{i_u}^{r \varpi_u^{(\sigma,1)}/2} \right) \\ & \quad \times \lambda_f \left(\prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (2,0)}} \widehat{p}_{i_u}^{r \varpi_u^{(\sigma,1)}/2} \prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (1,0)}} \widehat{p}_{i_u}^{r \varpi_u^{(\sigma,1)}} \right. \\ & \quad \left. \times \prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (0,1)}} \widehat{p}_{i_u}^{2j_{i_u} \varpi_u^{(\sigma,2)}} \right) \end{aligned}$$

where the two integers appearing in the right-hand side of the previous equality are different according to (6.25). Consequently, Proposition 2.2 and Lemmas 3.10 and 3.9 enable us to assert that

$$\begin{aligned} & \mathbb{E}_q^h \left(\prod_{u=1}^s \left(\lambda_f(\widehat{p}_{i_u}^r) \varpi_u^{(\sigma,1)} \prod_{j=1}^r \left(\lambda_f(\widehat{p}_{i_u}^{2j}) \varpi_{u,j}^{(\sigma,2)} \right) \right) \right) \\ & \ll \frac{1}{q} \prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (2,0)}} \frac{\log \widehat{p}_{i_u}}{\widehat{p}_{i_u}^{-r \varpi_u^{(\sigma,1)}/4}} \prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (1,0)}} \frac{\log \widehat{p}_{i_u}}{\widehat{p}_{i_u}^{-r \varpi_u^{(\sigma,1)}/4}} \\ & \quad \times \prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (0,1)}} \frac{\log \widehat{p}_{i_u}}{\widehat{p}_{i_u}^{-r \varpi_u^{(\sigma,2)}/2}}. \end{aligned}$$

Note that, in this first case, the right hand term is

$$\frac{1}{q} \prod_{u=1}^s \frac{\log \widehat{p}_{i_u}}{\widehat{p}_{i_u}^{-r(\varpi_u^{(\sigma,1)}/4 + \varpi_u^{(\sigma,2)}/2)},}$$

hence the contribution of these σ 's to $\mathbb{E}_q^h \left(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell \right)$ is bounded by

$$\frac{q^\varepsilon}{q} \left(\sum_{p \leq q^{vr}} \frac{1}{p^{1/2-r/4}} \right)^{m-\ell} \left(\sum_{p \leq q^{vr/2}} \frac{1}{p^{1-r/2}} \right)^\ell \ll q^{vr/4[(m-\ell)(r+2) + \ell r] - 1 + \varepsilon}.$$

This is an admissible error term as long as $vr/4[(m-\ell)(r+2) + \ell r] < 1$.

Second case. $\exists u_\sigma \in \{1, \dots, s\}$, $\varpi_{u_\sigma}^{(\sigma,1)}/2 + \varpi_{u_\sigma}^{(\sigma,2)} > 1$.

According to (2.11) and (2.12), if $1 \leq u \leq s$ and $1 \leq j \leq r$, then

$$\lambda_f(\widehat{p}_{i_u}^r)^{\varpi_u^{(\sigma,1)}} = \sum_{k_{u,1}=0}^{r\varpi_u^{(\sigma,1)}} x(\varpi_u^{(\sigma,1)}, r, k_{u,1}) \lambda_f(\widehat{p}_{i_u}^{k_{u,1}})$$

and

$$\lambda_f(\widehat{p}_{i_u}^{2j})^{\varpi_{u,j}^{(\sigma,2)}} = \sum_{k_{u,j,2}=0}^{j\varpi_{u,j}^{(\sigma,2)}} x(\varpi_{u,j}^{(\sigma,2)}, 2j, 2k_{u,j,2}) \lambda_f(\widehat{p}_{i_u}^{2k_{u,j,2}})$$

since $x(\varpi_{u,j}^{(\sigma,2)}, 2j, k_{u,j,2}) = 0$ if $k_{u,j,2}$ is odd (see (2.14)). Then, one may remark that

$$\prod_{1 \leq j \leq r} \lambda_f(\widehat{p}_{i_u}^{2k_{u,j,2}}) = \sum_{\ell_u=0}^{K_u} y_{\ell_u} \lambda_f(\widehat{p}_{i_u}^{2\ell_u})$$

for some integers y_{ℓ_u} and where $K_u := \sum_{1 \leq j \leq r} k_{u,j,2}$ for any $1 \leq u \leq s$. All these facts lead to

$$\begin{aligned} & \mathbb{E}_q^h \left(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell \right) \\ &= \frac{(-2)^m (-1)^{\ell r}}{\log^m(q^r)} \sum_{1 \leq j_1, \dots, j_\alpha \leq r} (-1)^{j_1 + \dots + j_\ell} \\ & \quad \times \sum_{s=1}^m \sum_{\substack{\sigma \in P(m,s) \\ \text{distinct}}} \sum_{i_1, \dots, i_s} \prod_{u=1}^s \left(\frac{\log \varpi_u^{(\sigma,1)} + \varpi_u^{(\sigma,2)}(\widehat{p}_{i_u})}{\widehat{p}_{i_u}^{\varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)}}} \right) \\ & \quad \times \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right)^{\varpi_u^{(\sigma,1)}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{i_u}}{\log(q^r)} \right)^{\varpi_u^{(\sigma,2)}} \end{aligned}$$

$$\begin{aligned} & \times \sum_{\substack{0 \leq k_{1,1} \leq r \varpi_1^{(\sigma,1)} \\ \vdots \\ 0 \leq k_{s,1} \leq r \varpi_s^{(\sigma,1)}}} \sum_{\substack{0 \leq k_{1,1,2} \leq \varpi_{1,1}^{(\sigma,2)} \\ \vdots \\ 0 \leq k_{s,1,2} \leq \varpi_{s,1}^{(\sigma,2)}}} \dots \sum_{\substack{0 \leq k_{1,r,2} \leq r \varpi_{1,r}^{(\sigma,2)} \\ \vdots \\ 0 \leq k_{s,r,2} \leq r \varpi_{s,r}^{(\sigma,2)}}} \sum_{\substack{0 \leq \ell_1 \leq K_1 \\ \vdots \\ 0 \leq \ell_s \leq K_s}} \\ & \times \prod_{u=1}^s \left(x \left(\varpi_u^{(\sigma,1)}, r, k_{u,1} \right) y_{\ell_u} \prod_{j=1}^r \left(x \left(\varpi_{u,j}^{(\sigma,2)}, 2j, 2k_{u,j,2} \right) \right) \right) \\ & \times \mathbb{E}_q^h \left(\lambda_f \left(\prod_{u=1}^s \widehat{p}_{i_u}^{k_{u,1}} \right) \lambda_f \left(\prod_{u=1}^s \widehat{p}_{i_u}^{2\ell_u} \right) \right). \end{aligned}$$

Proposition 2.2 and Lemmas 3.10 and 3.9 enable us to assert that

$$\begin{aligned} & \mathbb{E}_q^h \left(\lambda_f \left(\prod_{u=1}^s \widehat{p}_{i_u}^{k_{u,1}} \right) \lambda_f \left(\prod_{u=1}^s \widehat{p}_{i_u}^{2\ell_u} \right) \right) \\ & = \prod_{u=1}^s \delta_{k_{u,1}, 2\ell_u} + O \left(\frac{1}{q} \prod_{u=1}^s \widehat{p}_{i_u}^{k_{u,1}/4 + \ell_u/2} \log \widehat{p}_{i_u} \right) \end{aligned}$$

and we can write

$$\mathbb{E}_q^h \left(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell \right) = \text{TP} + O(\text{TE}) \tag{6.26}$$

with

$$\begin{aligned} \text{TP} & := \frac{(-2)^m (-1)^{\ell r}}{\log^m(q^r)} \sum_{1 \leq j_1, \dots, j_\ell \leq r} (-1)^{j_1 + \dots + j_\ell} \\ & \times \sum_{s=1}^{\alpha+m-\ell} \sum_{\substack{\sigma \in P(m,s) \\ i_1, \dots, i_s \\ \text{distinct}}} \sum_{u=1}^s \left(\frac{\log \varpi_u^{(\sigma,1)} + \varpi_u^{(\sigma,2)}}{\widehat{p}_{i_u}^{\varpi_u^{(\sigma,1)}/2} + \varpi_u^{(\sigma,2)}} \left(\widehat{p}_{i_u} \right) \right. \\ & \quad \times \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right)^{\varpi_u^{(\sigma,1)}} \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{i_u}}{\log(q^r)} \right)^{\varpi_u^{(\sigma,2)}} \left. \right) \\ & \times \sum_{\substack{0 \leq k_{1,1,2} \leq \varpi_{1,1}^{(\sigma,2)} \\ \vdots \\ 0 \leq k_{s,1,2} \leq \varpi_{s,1}^{(\sigma,2)}}} \dots \sum_{\substack{0 \leq k_{1,r,2} \leq r \varpi_{1,r}^{(\sigma,2)} \\ \vdots \\ 0 \leq k_{s,r,2} \leq r \varpi_{s,r}^{(\sigma,2)}}} \sum_{\substack{0 \leq \ell_1 \leq r \min(\varpi_1^{(\sigma,1)}/2, \varpi_1^{(\sigma,2)}) \\ \vdots \\ 0 \leq \ell_s \leq r \min(\varpi_s^{(\sigma,1)}/2, \varpi_s^{(\sigma,2)})}} \\ & \times \prod_{u=1}^s \left(x \left(\varpi_u^{(\sigma,1)}, r, 2\ell_u \right) y_{\ell_u} \prod_{j=1}^r \left(x \left(\varpi_{u,j}^{(\sigma,2)}, 2j, 2k_{u,j,2} \right) \right) \right) \tag{6.27} \end{aligned}$$

and

$$\begin{aligned} \text{TE} := & \frac{1}{q \log^m(q^r)} \sum_{s=1}^m \sum_{\sigma \in P(m,s)} \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{u=1}^s \left(\log^{\varpi_u^{(\sigma,1)} + \varpi_u^{(\sigma,2)} + 1}(\widehat{p}_{i_u}) \right. \\ & \times \widehat{p}_{i_u}^{(r/2-1)(\varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)})} \left| \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right|^{\varpi_u^{(\sigma,1)}} \\ & \times \left| \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{i_u}}{\log(q^r)} \right) \right|^{\varpi_u^{(\sigma,2)}} \Bigg), \end{aligned}$$

which is bounded by $O_\varepsilon(q^{mvr^2/4-1+\varepsilon})$ for any $\varepsilon > 0$ and is an admissible error term if $mv < 4/r^2$. Estimating TP is possible since we can assume that σ satisfies the following additional property. If $\varpi_u^{(\sigma,2)} = 0$ for some $1 \leq u \leq s$, then $\varpi_u^{(\sigma,1)} > 1$. Let us assume on the contrary that $\varpi_u^{(\sigma,1)} \leq 1$, which implies $\varpi_u^{(\sigma,1)} = 1$ according to (6.24). Then,

$$x(\varpi_u^{(\sigma,1)}, r, 2\ell_u) = x(1, r, 0) = 0$$

since $\ell_u = 0$ and according to (2.14). Thus, the contribution of the σ 's which do not satisfy this last property vanishes. As a consequence, the sum over the distinct i_1, \dots, i_s is bounded by

$$\begin{aligned} & \sum_{\substack{i_1, \dots, i_s \\ \text{distinct}}} \prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (2,0)}} \left(\frac{\log^2(\widehat{p}_{i_u})}{\widehat{p}_{i_u}} \left| \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right|^2 \right) \\ & \times \prod_{\substack{1 \leq u \leq s \\ (\varpi_u^{(\sigma,1)}, \varpi_u^{(\sigma,2)}) = (0,1)}} \left(\frac{\log(\widehat{p}_{i_u})}{\widehat{p}_{i_u}} \left| \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{i_u}}{\log(q^r)} \right) \right| \right) \\ & \times \prod_{\substack{1 \leq u \leq s \\ \varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)} > 1}} \left(\frac{\log^{\varpi_u^{(\sigma,1)} + \varpi_u^{(\sigma,2)}}(\widehat{p}_{i_u})}{\widehat{p}_{i_u}^{\varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)}}} \left| \widehat{\Phi} \left(\frac{\log \widehat{p}_{i_u}}{\log(q^r)} \right) \right|^{\varpi_u^{(\sigma,1)}} \right. \\ & \quad \left. \times \left| \widehat{\Phi} \left(\frac{2 \log \widehat{p}_{i_u}}{\log(q^r)} \right) \right|^{\varpi_u^{(\sigma,2)}} \right), \end{aligned}$$

which is itself bounded by $O(\log^{A_\sigma}(q))$ where the exponent is given by

$$\begin{aligned} A_\sigma := & 2\# \left\{ 1 \leq u \leq s : \varpi_u^{(\sigma,2)} = 0 \text{ and } \varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)} \leq 1 \right\} \\ & + \# \left\{ 1 \leq u \leq s : \varpi_u^{(\sigma,2)} = 1 \text{ and } \varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)} \leq 1 \right\} < m. \end{aligned}$$

The last inequality follows from (see (6.22) and the additional property of σ)

$$m = A_\sigma + \sum_{\substack{1 \leq u \leq s \\ \varpi_u^{(\sigma,1)}/2 + \varpi_u^{(\sigma,2)} > 1}} \left(\varpi_u^{(\sigma,1)} + \varpi_u^{(\sigma,2)} \right).$$

Thus, the contribution of the TP term of these σ 's to $\mathbb{E}_q^h(P_q^1[\Phi; r]^{m-\ell} P_q^2[\Phi; r]^\ell)$ is bounded by $O(\log^{-1}(q))$.

A Analytic and arithmetic toolbox

A.1 On smooth dyadic partitions of unity

Let $\psi: \mathbb{R}_+ \rightarrow \mathbb{R}$ be any smooth function satisfying

$$\psi(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > \sqrt{2}, \end{cases}$$

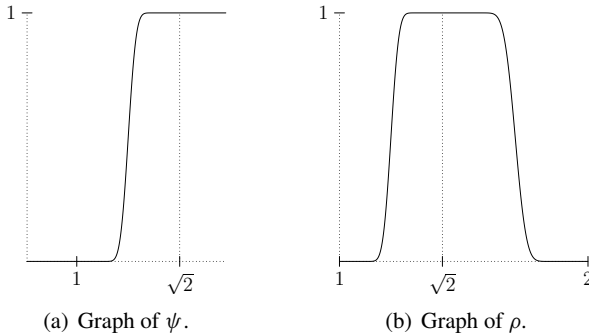
and $x^j \psi^{(j)}(x) \ll_j 1$ for any real number $x \geq 0$ and any integer $j \geq 0$. If $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}$ is the function defined by

$$\rho(x) := \begin{cases} \psi(x), & \text{if } 0 \leq x \leq \sqrt{2}, \\ 1 - \psi\left(\frac{x}{\sqrt{2}}\right), & \text{otherwise,} \end{cases}$$

then ρ is a smooth function compactly supported in $[1, 2]$ satisfying

$$x^j \rho^{(j)}(x) \ll_j 1 \quad \text{and} \quad \sum_{a \in \mathbb{Z}} \rho\left(\frac{x}{\sqrt{2}^a}\right) = 1$$

for any real number $x \geq 0$ and any integer $j \geq 0$.



If $F: \mathbb{R}_+^n \rightarrow \mathbb{R}$ is a function of $n \geq 1$ real variables, then we can decompose it in

$$F = \sum_{a_1 \in \mathbb{Z}} \dots \sum_{a_n \in \mathbb{Z}} F_{A_1, \dots, A_n}$$

where $A_i := \sqrt{2}^{a_i}$ and

$$F_{A_1, \dots, A_n}(x_1, \dots, x_n) := \prod_{i=1}^n \rho_{A_i}(x_i) F(x_1, \dots, x_n)$$

with $\rho_{A_i}(x_i) := \rho(x_i / A_i)$ a smooth function compactly supported in $[A_i, 2A_i]$ satisfying $x_i^j \rho_{A_i}^{(j)}(x_i) \ll_j 1$ for any real number $x_i \geq 0$ and any integer $j \geq 0$. Let us introduce the following notation for summation over powers of $\sqrt{2}$:

$$\sum_{A \leq M \leq B}^\# f(M) := \sum_{\substack{n \in \mathbb{N} \\ A \leq 2^{n/2} \leq B}} f(2^{n/2}).$$

We will use such smooth dyadic partitions of unity several times in this paper. We will also need the two following natural estimates:

$$\sum_{M \leq M_1}^\# M^\alpha \ll M_1^\alpha \tag{A.1}$$

for any $\alpha, M_1 > 0$ and

$$\sum_{M \geq M_0}^\# \frac{1}{M^\alpha} \ll \frac{1}{M_0^\alpha} \tag{A.2}$$

for any $\alpha, M_0 > 0$.

A.2 On Bessel functions

The Bessel function of the first kind and order an integer $\kappa \geq 1$ is defined by

$$\forall z \in \mathbb{C}, \quad J_\kappa(z) := \sum_{n \geq 0} \frac{(-1)^n}{n!(\kappa + n)!} \left(\frac{z}{2}\right)^{\kappa + 2n}.$$

It satisfies (see [25, Lemma C.2])

$$\left(\frac{x}{1+x}\right)^j J_\kappa^{(j)}(x) \ll_{j, \kappa} \frac{1}{(1+x)^{\frac{1}{2}}} \left(\frac{x}{1+x}\right)^\kappa \tag{A.3}$$

for any real number x , any integer $j \geq 0$ and any integer $\kappa \geq 1$. The next useful lemma follows immediately.

Lemma A.1. *Let $X > 0$ and $\kappa \geq 1$, then*

$$\sum_{d \geq 1} \frac{\tau(d)}{\sqrt{d}} \left| J_\kappa \left(\frac{X}{d} \right) \right| \ll \begin{cases} X^{1/2} \log X, & \text{if } X > 1, \\ X^\kappa, & \text{if } 0 < X \leq 1. \end{cases}$$

A.3 Basic facts on Kloosterman sums

For any integer $m, n, c \geq 1$, the Kloosterman sum is defined by

$$S(m, n; c) := \sum_{\substack{x \pmod{c} \\ (x, c) = 1}} e \left(\frac{mx + n\bar{x}}{c} \right)$$

where \bar{x} stands for the inverse of x modulo c . We recall some basic facts on these sums. The Chinese Remainder Theorem implies the following multiplicativity relation:

$$S(m, n; qr) = S(m\bar{q}^2, n; r)S(m\bar{r}^2, n; q), \tag{A.4}$$

valid when $(q, r) = 1$. Here, \bar{q} (resp. \bar{r}) is the inverse of q (resp. r) modulo r (resp. q). If p and q are two prime numbers, $\gamma \geq 1$ and $r \geq 1$, then from (A.4) and [7, (2.312)] we obtain

$$S(p^\gamma q, 1; qr) = \begin{cases} -S(p^\gamma \bar{q}, 1; r), & \text{if } (q, r) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{A.5}$$

The Weil–Estermann inequality [8] is

$$|S(m, n; c)| \leq \sqrt{(m, n, c)} \tau(c) \sqrt{c}. \tag{A.6}$$

Acknowledgments. This work began while both authors shared the hospitality of Centre de Recherches Mathématiques (Montréal) during the theme year “Analysis in Number Theory” (first semester of 2006). We would like to thank C. David, H. Darmon and A. Granville for their invitation. This work was essentially completed in September 2006 in CIRM (Luminy) at the occasion of J.-M. Deshouillers’ sixtieth birthday. We would like to wish him the best. Finally, we thank Emmanuel Kowalski, Steven J. Miller, Zeév Rudnick and the referees for very valuable comments.

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Received May 10, 2007; revised November 30, 2009.

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