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EVALUATING CONVOLUTION SUMS OF THE DIVISOR FUNCTION BY QUASIMODULAR FORMS

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We provide a systematic method to compute arithmetic sums including some previously computed by Alaca, Besge, Cheng, Glaisher, Huard, Lahiri, Lemire, Melfi, Ou, Ramanujan, Spearman and Williams. Our method is based on quasimodular forms. This extension of modular forms has been constructed by Kaneko and Zagier.

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1. Introduction

1.1. *Results*

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{N}^* = \mathbb{N}\setminus\{0\}$. For n and j in $\mathbb{N}^*$ we set

$$\sigma_j(n) := \sum_{d|n} d^j$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}^*$ we set $\sigma_j(n) = 0$. Following [27], for $N \in \mathbb{N}^*$ we define

$$W_N(n) := \sum_{m<n/N} \sigma_1(m)\sigma_1(n - Nm)$$

where m runs through the positive integers $<n/N$. We call W_N the convolution of level N (of the divisor function). We present a method (introduced in [17]) to compute some of these sums using quasimodular forms. We insist on the fact that the only goal of this paper is to present a method and we recapitulate, in Table 1 some of the known results. We hope that some our results are new (see e.g. Theorem 1.3 and Proposition 1.13). The evaluations of $W_N(n)$ for $N \in \{1, 2, 3, 4\}$ given in [12] are elementary and the ones of $W_N(n)$ for $N \in \{5, \dots, 9\}$ are analytic in nature and use the ideas of Ramanujan. Our evaluations are on algebraic nature.

Table 1. Some previous computations of W_N.

Level N	Who	Where
1	Besge (Liouville), Glaisher, Ramanujan	[6, 11, 21]
2, 3, 4	Huard, Ou, Spearman & Williams	[12]
5, 7	Lemire & Williams	[18]
6	Alaca & Williams	[5]
8	Williams	[28]
9	Williams	[27]
12	Alaca, Alaca & Williams	[2]
16	Alaca, Alaca & Williams	[1]
18	Alaca, Alaca & Williams	[4]
24	Alaca, Alaca & Williams	[3]

For $N \in [5, 10]$, we denote by $\Delta_{4,N}$ the unique cuspidal form spanning the cuspidal subspace of the modular forms of weight 4 on $\Gamma_0(N)$ with Fourier expansion[a] $\Delta_{4,N}(z) = e^{2\pi i z} + O(e^{4\pi i z})$. We define

$$\Delta_{4,N}(z) =: \sum_{n=1}^{+\infty} \tau_{4,N}(n) e^{2\pi i n z}.$$

We also write

$$\begin{aligned}\Delta(z) &:= e^{2i\pi z} \prod_{n=1}^{+\infty} [1 - e^{2\pi i n z}]^{24} \\ &=: \sum_{n=1}^{+\infty} \tau(n) e^{2\pi i n z}\end{aligned}$$

for the unique primitive form of weight 12 on $\mathrm{SL}(2, \mathbb{Z})$.

Theorem 1.1. *Let* $n \in \mathbb{N}^*$, *then*

$$\begin{aligned}
W_1(n) &= \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n), \\
W_2(n) &= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right), \\
W_3(n) &= \frac{1}{24}\sigma_3(n) + \frac{3}{8}\sigma_3\left(\frac{n}{3}\right) - \frac{1}{12}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{3}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{3}\right), \\
W_4(n) &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right) \\
&\quad + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right),
\end{aligned}$$

[a]In this paper, "Fourier expansion" always means "Fourier expansion at the cusp ∞".

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) - \frac{1}{20}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{5}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{5}\right) - \frac{1}{130}\tau_{4,5}(n),$$

$$W_6(n) = \frac{1}{120}\sigma_3(n) + \frac{1}{30}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{40}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{6}\right) - \frac{1}{24}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{6}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{6}\right) - \frac{1}{120}\tau_{4,6}(n),$$

$$W_7(n) = \frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\left(\frac{n}{7}\right) - \frac{1}{28}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{7}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{7}\right) - \frac{1}{70}\tau_{4,7}(n),$$

$$W_8(n) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) - \frac{1}{32}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{8}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{8}\right) - \frac{1}{64}\tau_{4,8}(n),$$

$$W_9(n) = \frac{1}{216}\sigma_3(n) + \frac{1}{27}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{8}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{36}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{9}\right) - \frac{1}{54}\tau_{4,9}(n).$$

$$W_{10}(n) = \frac{1}{312}\sigma_3(n) + \frac{1}{78}\sigma_3\left(\frac{n}{2}\right) + \frac{25}{312}\sigma_3\left(\frac{n}{5}\right) + \frac{25}{78}\sigma_3\left(\frac{n}{10}\right) - \frac{1}{40}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{10}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{10}\right) - \frac{1}{120}\tau_{4,10}(n) - \frac{3}{260}\tau_{4,5}(n) - \frac{3}{65}\tau_{4,5}\left(\frac{n}{2}\right).$$

For dimensional reasons, the forms $\Delta_{4,N}$ are primitive forms for $N \in \{5, \dots, 10\}$. It follows that the arithmetic functions $\tau_{4,N}$ are multiplicative and satisfy the relation (1.2) (see below). Following [14], one obtains

$$\Delta_{4,5}(z) = [\Delta(z)\Delta(5z)]^{1/6},$$
$$\Delta_{4,6}(z) = [\Delta(z)\Delta(2z)\Delta(3z)\Delta(6z)]^{1/12},$$
$$\Delta_{4,8}(z) = [\Delta(2z)\Delta(4z)]^{1/6},$$
$$\Delta_{4,9}(z) = [\Delta(3z)]^{1/3},$$

whereas $\Delta_{4,7}$ and $\Delta_{4,10}$ *are not* products of the Δ function. However, using MAGMA [7] (see [26] for the algorithms based on the computation of the spectrum of Hecke operators on modular symbols), one can compute their Fourier coefficients

Table 2. First Fourier coefficients of $\Delta_{4,7}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,7}(n)$	1	−1	−2	−7	16	2	−7	15	−23	−16	−8
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,7}(n)$	14	28	7	−32	41	54	23	−110	−112	14	8

Table 3. First Fourier coefficients of $\Delta_{4,10}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,10}(n)$	1	2	−8	4	5	−16	−4	8	37	10	12
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,10}(n)$	−32	−58	−8	−40	16	66	74	−100	20	32	24

(see Tables 2 and 3).

Remark 1.2. The independent computation of W_7 by Lemire and Williams [18] implies that

$$\Delta_{4,7}(z) = [(\Delta(z)^2\Delta(7z))^{1/3} + 13(\Delta(z)\Delta(7z))^{1/2} + 49(\Delta(z)\Delta(7z)^2)^{1/3}]^{1/3}.$$

It is likely that, following [18] to evaluate W_{10} we could get an expression of $\Delta_{4,10}$.

In each of our previous examples, we did not leave the field of rational numbers. This might not happen, since the primitive forms do not necessarily have rational coefficients. However, every evaluation will make use of totally real algebraic numbers for coefficients since the extension of $\mathbb{Q}$ by the Fourier coefficients of a primitive form is finite and totally real [24, Proposition 1.3]. To illustrate that fact, we shall evaluate the convolution sum of levels 11 and 13. The set of primitive modular forms of weight 4 on $\Gamma_0(11)$ has two elements. The coefficients of these two primitive forms are in $\mathbb{Q}(t)$ where t is a root of $X^2 - 2X - 2$ (see Sec. 2.8 for the use of a method founded in [29]). Each primitive form is determined by the beginning of its Fourier expansion:

$$\Delta_{4,11,1}(z) = e^{2\pi iz} + (2-t)e^{4\pi iz} + O(e^{6\pi iz}),$$
$$\Delta_{4,11,2}(z) = e^{2\pi iz} + te^{4\pi iz} + O(e^{6\pi iz}).$$

We denote by $\tau_{4,11,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{4,11,i}$. The two primitive forms, and hence their Fourier coefficients, are conjugate by $t \mapsto 2-t$ (see e.g. [9] for the general result and Sec. 2.8 for the special case needed here).

Theorem 1.3. *Let $n \in \mathbb{N}^*$. Then*

$$W_{11}(n) = \frac{5}{1464}\sigma_3(n) + \frac{605}{1464}\sigma_3\left(\frac{n}{11}\right) - \frac{2t+43}{4026}\tau_{4,11,1}(n) + \frac{2t-47}{4026}\tau_{4,11,2}(n)$$
$$- \frac{1}{44}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{11}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{11}\right).$$

Table 4. First Fourier coefficients of $\Delta_{4,11,1}$ where $t^2 - 2t - 2 = 0$.

n	1	2	3	4	5
$\tau_{4,11,1}(n)$	1	$-t+2$	$4t-5$	$-2t-2$	$-8t+9$
n	6	7	8	9	10
$\tau_{4,11,1}(n)$	$5t-18$	$4t+6$	$10t-16$	$-8t+30$	$-9t+34$
n	11	12	13	14	15
$\tau_{4,11,1}(n)$	-11	$-14t-6$	$20t+20$	$-6t+4$	$12t-109$

Table 5. First Fourier coefficients of $\Delta_{4,13,2}$ where $u^2 - u - 4 = 0$.

n	1	2	3	4	5
$\tau_{4,13,2}(n)$	1	$-u+1$	$3u+1$	$-u-3$	$-u-1$
n	6	7	8	9	10
$\tau_{4,13,2}(n)$	$-u-11$	$-11u+1$	$11u-7$	$15u+10$	$u+3$
n	11	12	13	14	15
$\tau_{4,13,2}(n)$	$-12u+46$	$-13u-15$	-13	$-u+45$	$-7u-13$

Remark 1.4. We have

$$-\frac{2t+43}{4026}\tau_{4,11,1}(n) + \frac{2t-47}{4026}\tau_{4,11,2}(n) = \mathrm{tr}_{\mathbb{Q}(t)/\mathbb{Q}}\left[-\frac{2t+43}{4026}\tau_{4,11,1}(n)\right] \in \mathbb{Q}.$$

The set of primitive modular forms of weight 4 on $\Gamma_0(13)$ has three elements. One of them, we note $\Delta_{4,13,1}$ has Fourier coefficients in $\mathbb{Q}$. The two others, we note $\Delta_{4,13,2}$ and $\Delta_{4,13,3}$, have Fourier coefficients in $\mathbb{Q}(u)$ where u is a root of $X^2 - X - 4$. Each of these two primitive form is determined by the beginning of its Fourier expansion:

$$\Delta_{4,13,2}(z) = e^{2\pi iz} + (1-u)e^{4\pi iz} + O(e^{6\pi iz}),$$
$$\Delta_{4,13,3}(z) = e^{2\pi iz} + ue^{4\pi iz} + O(e^{6\pi iz}).$$

We denote by $\tau_{4,13,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{4,13,i}$. The two primitive forms $\Delta_{4,13,2}$ and $\Delta_{4,12,3}$, and hence their Fourier coefficients, are conjugate by $t \mapsto 1-t$ (see e.g. [9]).

Theorem 1.5. *Let $n \in \mathbb{N}^*$. Then*

$$W_{13}(n) = \frac{1}{408}\sigma_3(n) + \frac{169}{408}\sigma_3\left(\frac{n}{13}\right) + \frac{u-6}{442}\tau_{4,13,2}(n) - \frac{u+5}{442}\tau_{4,13,3}(n)$$
$$- \frac{1}{52}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{13}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{13}\right).$$

Remark 1.6. We have

$$\frac{u-6}{442}\tau_{4,13,2}(n) - \frac{u+5}{442}\tau_{4,13,3}(n) = \mathrm{tr}_{\mathbb{Q}(u)/\mathbb{Q}}\left[\frac{u-6}{442}\tau_{4,11,2}(n)\right] \in \mathbb{Q}.$$

The set of primitive modular forms of weight 4 on $\Gamma_0(14)$ has two elements. Both have coefficients in $\mathbb{Q}$ and we can distinguish them by the beginning of their Fourier expansion:

$$\Delta_{4,14,1}(z) = e^{2\pi iz} + 2e^{4\pi iz} + O(e^{6\pi iz}),$$
$$\Delta_{4,14,2}(z) = e^{2\pi iz} - 2e^{4\pi iz} + O(e^{6\pi iz}).$$

We denote by $\tau_{4,14,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{4,14,i}$ and give in Sec. 2.10 a method to compute these coefficients and get Tables 6 and 7.

Theorem 1.7. *Let $n \in \mathbb{N}^*$. Then*

$$W_{14}(n) = \frac{1}{600}\sigma_3(n) + \frac{1}{150}\sigma_3\left(\frac{n}{2}\right) + \frac{49}{600}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{150}\sigma_3\left(\frac{n}{14}\right) - \frac{1}{56}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{14}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{14}\right) - \frac{3}{350}\tau_{4,7}(n) - \frac{6}{175}\tau_{4,7}\left(\frac{n}{2}\right) - \frac{1}{84}\tau_{4,14,1}(n) - \frac{1}{200}\tau_{4,14,2}(n).$$

Remark 1.8. The fact that for each $N \in \{12, 16, 18, 24\}$ there exists only one primitive form of weight 4 over $\Gamma_0(N)$ and at least one of weight 2 implies that the only modular forms appearing in the evaluation of the corresponding W_N have rational coefficients.

Our method, with the introduction of Dirichlet characters, also allows to recover a second result of Williams [27, Theorem 1.2] which extended a result of Melfi [20, Theorem 2, (7)]. This result is Theorem 1.9. For $b \in \mathbb{N}^*$ and $a \in \{0, \ldots, b-1\}$, we define

$$S[a,b](n) := \sum_{\substack{m=0 \\ m \equiv a \pmod{b}}}^{n} \sigma_1(m)\sigma_1(n-m).$$

Table 6. First Fourier coefficients of $\Delta_{4,14,1}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,14,1}(n)$	1	2	−2	4	−12	−4	7	8	−23	−24	48
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,14,1}(n)$	−8	56	14	24	16	−114	−46	2	−48	−14	96

Table 7. First Fourier coefficients of $\Delta_{4,14,2}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,14,2}(n)$	1	−2	8	4	−14	−16	−7	−8	37	28	−28
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,14,2}(n)$	32	18	14	−112	16	74	−74	80	−56	−56	56

We compute $S[i,3]$ for $i \in \{0,1,2\}$. Our result uses the primitive Dirichlet character χ_3 defined by

$$\chi_3(n) := \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 1 \pmod 3 \\ -1 & \text{if } n \equiv -1 \pmod 3 \end{cases}$$

for all $n \in \mathbb{N}^*$.

Theorem 1.9. *Let $n \in \mathbb{N}^*$, then*

$$S[0,3](n) = \frac{11}{72}\sigma_3(n) + \frac{25}{18}\sigma_3\left(\frac{n}{3}\right) - \frac{9}{8}\sigma_3\left(\frac{n}{9}\right) - \frac{1}{4}n\sigma_1(n) - n\sigma_1\left(\frac{n}{3}\right) + \frac{3}{4}n\sigma_1\left(\frac{n}{9}\right)$$
$$+ \frac{1}{24}[1 + \delta(3|n)]\sigma_1(n) + \frac{1}{18}\tau_{4,9}(n),$$

$$S[1,3](n) = \frac{19}{144}\sigma_3(n) + \frac{1}{48}\chi_3(n)\sigma_3(n) - \frac{25}{36}\sigma_3\left(\frac{n}{3}\right) + \frac{9}{16}\sigma_3\left(\frac{n}{9}\right)$$
$$- \frac{1}{8}n\sigma_1(n) - \frac{1}{8}\chi_3(n)n\sigma_1(n) + \frac{1}{2}n\sigma_1\left(\frac{n}{3}\right) + \frac{3}{8}\chi_3(n)n\sigma_1\left(\frac{n}{3}\right)$$
$$- \frac{3}{8}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\delta(3|n-1)\sigma_1(n) + \frac{1}{18}\tau_{4,9}(n),$$

and

$$S[2,3](n) = \frac{19}{144}\sigma_3(n) - \frac{1}{48}\chi_3(n)\sigma_3(n) - \frac{25}{36}\sigma_3\left(\frac{n}{3}\right) + \frac{9}{16}\sigma_3\left(\frac{n}{9}\right)$$
$$- \frac{1}{8}n\sigma_1(n) + \frac{1}{8}\chi_3(n)n\sigma_1(n) + \frac{1}{2}n\sigma_1\left(\frac{n}{3}\right) - \frac{3}{8}\chi_3(n)n\sigma_1\left(\frac{n}{3}\right)$$
$$- \frac{3}{8}n\sigma_1\left(\frac{n}{9}\right) + \frac{1}{24}\delta(3|n-2)\sigma_1(n) - \frac{1}{9}\tau_{4,9}(n),$$

where $\delta(3|n)$ is 1 if 3 divides n and 0 otherwise.

We next consider convolutions of different divisor sums and recover results of Melfi [20, Theorem 2, (9, 10)] completed by Huard, Ou, Spearman and Williams [12, Theorem 6] and Cheng and Williams [8]. We shall use the unique cuspidal form $\Delta_{8,2}$ spanning the cuspidal subspace of the modular forms of weight 8 on $\Gamma_0(2)$ with Fourier expansion $\Delta_{8,2}(z) = e^{2\pi iz} + O(e^{4\pi iz})$. Using [14], we have

$$\Delta_{8,2}(z) = [\Delta(z)\Delta(2z)]^{1/3}.$$

We define

$$\Delta_{8,2}(z) = \sum_{n=1}^{+\infty} \tau_{8,2}(n)e^{2\pi inz}.$$

This is again a primitive form, hence the arithmetic function $\tau_{8,2}$ is multiplicative and satisfies the relation (1.2) (see below).

Theorem 1.10. *Let $n \in \mathbb{N}^*$. Then*

$$\sum_{k=0}^{n} \sigma_1(k)\sigma_3(n-k) = \frac{7}{80}\sigma_5(n) - \frac{1}{8}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(n-2k)\sigma_3(k) = \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_3\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(k)\sigma_3(n-2k) = \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5\left(\frac{n}{2}\right) - \frac{1}{16}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma_1\left(\frac{n}{2}\right).$$

Moreover,

$$\sum_{k=0}^{n} \sigma_1(k)\sigma_5(n-k) = \frac{5}{126}\sigma_7(n) - \frac{1}{12}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n),$$

$$\sum_{k<n/2} \sigma_1(k)\sigma_5(n-2k) = \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{24}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{1}{504}\sigma_1\left(\frac{n}{2}\right) - \frac{1}{102}\tau_{8,2}(n),$$

and

$$\sum_{k<n/2} \sigma_1(n-2k)\sigma_5(k) = \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{12}n\sigma_5\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}\tau_{8,2}(n).$$

In Theorem 1.10, the first and fourth identities are due to Ramanujan [21]. The second and third ones are due to Huard, Ou, Spearman and Williams [12, Theorem 6]. The fifth and sixth ones are due to Cheng and Williams [8]. Some other identities of the same type may be found in [8, 21].

Our method also allows to evaluate sums of Lahiri type

$$S[(a_1,\dots,a_r),(b_1,\dots,b_r),(N_1,\dots,N_r)](n) := \sum_{\substack{(m_1,\dots,m_r)\in\mathbb{N}^r\\ m_1+\dots+m_r=n}} m_1^{a_1}\cdots m_r^{a_r}\sigma_{b_1}\left(\frac{m_1}{N_1}\right)\cdots\sigma_{b_r}\left(\frac{m_r}{N_r}\right) \qquad (1.1)$$

[15, 16] and [12, Sec. 3] where the a_i are nonnegative integers, the N_i are positive integers and the b_i are odd positive integers. To simplify the notations, we introduce

$$S[(a_1, \dots, a_r), (b_1, \dots, b_r)](n) := S[(a_1, \dots, a_r), (b_1, \dots, b_r), (1, \dots, 1)](n).$$

For example, we prove the following:

Theorem 1.11. *Let $n \in \mathbb{N}^*$. Then*

$$\begin{aligned} S[(0,1,1),(1,1,1)](n) &= \frac{1}{288}n^2\sigma_5(n) - \frac{1}{72}n^3\sigma_3(n) + \frac{1}{288}n^2\sigma_3(n) \\ &\quad + \frac{1}{96}n^4\sigma_1(n) - \frac{1}{288}n^3\sigma_1(n). \end{aligned}$$

and

$$\begin{aligned} &-24^5 S[(0,0,0,1,1),(1,1,1,1,1)](n) \\ &\quad = -\frac{48}{5}n^2\sigma_9(n) + 128n^3\sigma_7(n) - 80n^2\sigma_7(n) - 600n^4\sigma_5(n) \\ &\quad + 648n^3\sigma_5(n) + \frac{8208}{7}n^5\sigma_3(n) - 144n^2\sigma_5(n) - \frac{11232}{7}n^4\sigma_3(n) - \frac{3456}{5}n^6\sigma_1(n) \\ &\quad + 576n^3\sigma_3(n) + \frac{5184}{5}n^5\sigma_1(n) - 432n^4\sigma_1(n) - 48n^2\sigma_3(n) \\ &\quad + 48n^3\sigma_1(n) + \frac{8}{35}n\tau(n) - \frac{8}{35}\tau(n). \end{aligned}$$

The first identity of Theorem 1.11 is due to Lahiri [15, (5.9)] and an elementary proof had been given by Huard, Ou, Spearman and Williams [12]. The second identity is due to Lahiri [16].

We continue our evaluations by the more complicated sum $S[(0,1),(1,1),(2,5)]$. The reason why it is more difficult is that the underlying space of new cuspidal modular forms has dimension 3.

The space of newforms of weight 6 on $\Gamma_0(10)$ has dimension 3. Let $\{\Delta_{6,10,i}\}_{1\le i\le 3}$ be the unique basis of primitive forms with

$$\begin{aligned} \Delta_{6,10,1}(z) &= e^{2\pi iz} + 4e^{4\pi iz} + 6e^{6\pi iz} + O(e^{8\pi iz}), \\ \Delta_{6,10,2}(z) &= e^{2\pi iz} - 4e^{4\pi iz} + 24e^{6\pi iz} + O(e^{8\pi iz}), \\ \Delta_{6,10,3}(z) &= e^{2\pi iz} - 4e^{4\pi iz} - 26e^{6\pi iz} + O(e^{8\pi iz}). \end{aligned}$$

Again, by [14], we know that these functions *are not* products of the Δ function. We denote by $\tau_{6,10,i}(n)$ the nth Fourier coefficient of $\Delta_{6,10,i}$. Note that the sequences $\tau_{6,10,i}$ are multiplicative. Again, Stein's algorithms on Magma give the following tables.

We also need the unique primitive form

$$\Delta_{6,5}(z) = \sum_{n=1}^{+\infty} \tau_{6,5}(n) e^{2\pi inz}$$

Table 8. First Fourier coefficients of $\Delta_{6,10,1}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,1}(n)$	1	4	6	16	−25	24	−118	64	−207	−100	192
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,1}(n)$	96	1106	−472	−150	256	762	−828	−2740	−400	−708	768

Table 9. First Fourier coefficients of $\Delta_{6,10,2}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,2}(n)$	1	−4	24	16	25	−96	−172	−64	333	−100	132
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,2}(n)$	384	−946	688	600	256	−222	−1332	500	400	−4128	−528

Table 10. First Fourier coefficients of $\Delta_{6,10,3}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,10,3}(n)$	1	−4	−26	16	−25	104	−22	−64	433	100	−768
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,10,3}(n)$	−416	−46	88	650	256	378	−1732	1100	−400	572	3072

Table 11. First Fourier coefficients of $\Delta_{6,5}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{6,5}(n)$	1	2	−4	−28	25	−8	192	−120	−227	50	−148
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{6,5}(n)$	112	286	384	−100	656	−1678	−454	1060	−700	−768	−296

of weight 6 on $\Gamma_0(5)$. It is not a product of the Δ function, and its first Fourier coefficients are given in the following table.

Proposition 1.12. *Let $n \in \mathbb{N}^*$. Define*

$$A(n) = \frac{12}{13} n\sigma_3(n) + \frac{48}{13} n\sigma_3\left(\frac{n}{2}\right) + \frac{300}{13} n\sigma_3\left(\frac{n}{5}\right) + \frac{1200}{13} n\sigma_3\left(\frac{n}{10}\right),$$

$$B(n) = -\frac{48}{5} n^2\sigma_1\left(\frac{n}{2}\right) - 48n^2\sigma_1\left(\frac{n}{5}\right),$$

$$C(n) = 24n\sigma_1\left(\frac{n}{5}\right),$$

$$D(n) = \frac{12}{5} n\tau_{4,10}(n) - \frac{216}{65} n\tau_{4,5}(n) - \frac{864}{65} n\tau_{4,5}\left(\frac{n}{2}\right),$$

$$E(n) = \frac{108}{35} \tau_{6,5}(n) + \frac{864}{35} \tau_{6,5}\left(\frac{n}{2}\right),$$

$$F(n) = -\frac{24}{5} \tau_{6,10,1}(n) + \frac{12}{7} \tau_{6,10,2}(n).$$

Table 12. First Fourier coefficients of $\Delta_{8,5,1}$.

n	1	2	3	4	5	6	7	8	9
$\tau_{8,5,1}(n)$	1	−14	−48	68	125	672	−1644	840	117
n	10	11	12	13	14	15	16	17	18
$\tau_{8,5,1}(n)$	−1750	172	−3264	3862	23016	−6000	−20464	−12254	−1638

Table 13. First Fourier coefficients of $\Delta_{8,5,2}$ where $t^2 - 20t + 24 = 0$.

n	1	2	3	4	5
$\tau_{8,5,2}(n)$	1	$-t+20$	$8t-70$	$-20t+248$	-125
n	6	7	8	9	10
$\tau_{8,5,2}(n)$	$70t-1208$	$-56t+510$	$-120t+1920$	$160t+1177$	$125t-2500$
n	11	12	13	14	15
$\tau_{8,5,2}(n)$	$-400t+6272$	$184t-13520$	$608t-4310$	$-510t+8856$	$-1000t+8750$

Then

$$5 \times 24^2 \sum_{\substack{(a,b)\in\mathbb{N}^{*2} \\ 2a+5b=n}} b\sigma_1(a)\sigma_1(b) = A(n) + B(n) + C(n) + D(n) + E(n) + F(n).$$

We shall now evaluate $S[(1,1),(1,1),(1,5)]$ since it constitutes an example leaving the rational field. Let t be one of the two roots of $X^2 - 20X + 24$. There exist three primitive forms of weight 8 on $\Gamma_0(5)$ determined by the beginning of their Fourier expansion:

$$\begin{aligned}
\Delta_{8,5,1}(z) &= e^{2\pi iz} - 14e^{4\pi iz} + O(e^{6\pi iz}),\\
\Delta_{8,5,2}(z) &= e^{2\pi iz} + (20-t)e^{4\pi iz} + O(e^{6\pi iz}),\\
\Delta_{8,5,3}(z) &= e^{2\pi iz} + te^{4\pi iz} + O(e^{6\pi iz}).
\end{aligned}$$

The function $\Delta_{8,5,3}$ is obtained from $\Delta_{8,5,2}$ by the conjugation $(t \mapsto 20-t)$ of $\mathbb{Q}(t)$ on the Fourier coefficients. We denote by $\tau_{8,5,i}$ the multiplicative function given by the Fourier coefficients of $\Delta_{8,5,i}$.

Proposition 1.13. *Let $n \in \mathbb{N}^*$. Define*

$$\begin{aligned}
A(n) &= \frac{24}{13}n^2\sigma_3(n) + \frac{600}{13}n^2\sigma_3\left(\frac{n}{5}\right),\\
B(n) &= -\frac{24}{5}n^3\sigma_1(n) - 24n^3\sigma_1\left(\frac{n}{5}\right),\\
C(n) &= -\frac{288}{325}n^2\tau_{4,5}(n),\\
D(n) &= \frac{792+12t}{475}\tau_{8,5,2}(n) + \frac{1032-12t}{475}\tau_{8,5,3}(n).
\end{aligned}$$

Then

$$5 \times 24^2 \sum_{\substack{(a,b)\in\mathbb{N}^{*2}\\ a+5b=n}} ab\sigma_1(a)\sigma_1(b) = A(n) + B(n) + C(n) + D(n).$$

Remark 1.14. The two terms in the right-hand side of the definition of $D(n)$ in Proposition 1.13 being conjugate, we have

$$D(n) = \mathrm{tr}_{\mathbb{Q}(t)/\mathbb{Q}}\left[\frac{792 + 12t}{475}\tau_{8,5,2}(n)\right] \in \mathbb{Q}.$$

To stay in the field of rational numbers, we could have used the fundamental fact that, for every even $k > 0$ and every integer $N \geq 1$, the space of cuspidal forms of weight k on $\Gamma_0(N)$ has a basis whose elements have a Fourier expansion with integer coefficients [25, Theorem 3.52]. However, the coefficients of these Fourier expansions are often not multiplicative: this is a good reason to leave $\mathbb{Q}$.

Remark 1.15. If τ_* is one of our τ functions, its values are the Fourier coefficients of a primitive form (of weight k on $\Gamma_0(N)$ say). It therefore satisfies the following multiplicativity relation

$$\tau_*(mn) = \sum_{\substack{d|(m,n)\\ (d,N)=1}} \mu(d)d^{k-1}\tau_*\left(\frac{m}{d}\right)\tau_*\left(\frac{n}{d}\right). \tag{1.2}$$

1.2. *Method*

Since our method is based on quasimodular forms (anticipated by Rankin [22] and formally introduced by Kaneko and Zagier in [13]), we briefly recall the basics on these functions, referring to [17, 19] for the details.

Define

$$\Gamma_0(N) = \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} : (a,b,c,d) \in \mathbb{Z}^4, ad - bc = 1, N|c\right\}$$

for all integers $N \geq 1$. In particular, $\Gamma_0(1)$ is $\mathrm{SL}(2,\mathbb{Z})$. Denote by $\mathscr{H}$ the Poincaré upper half plane:

$$\mathscr{H} = \{z \in \mathbb{C}\colon \Im m\, z > 0\}.$$

Definition 1.16. Let $N \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $s \in \mathbb{N}^*$. A holomorphic function

$$f\colon \mathscr{H} \to \mathbb{C}$$

is a quasimodular form of weight k, depth s on $\Gamma_0(N)$ if there exist holomorphic functions $f_0, f_1, \ldots, f_s$ on $\mathscr{H}$ such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{i=0}^{s} f_i(z)\left(\frac{c}{cz+d}\right)^i \tag{1.3}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ and bounded by a positive power of $(|z|^2+1)/y$. By convention, the 0 function is a quasimodular form of depth 0 for each weight.

One can show [19, Lemma 119] that if f satisfies the quasimodularity condition (1.3), then f_s satisfies the modularity condition

$$(cz+d)^{-(k-2s)} f_s\left(\frac{az+b}{cz+d}\right) = f_s(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. The growth condition on f implies that f_s is a modular form of weight $k-2s$. Hence, if f is a quasimodular form of weight k and depth s, non identically vanishing, then k is even and $s \leq k/2$.

A fundamental quasimodular form is the Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) e^{2\pi i n z}.$$

It is a quasimodular form of weight 2, depth 1 on $\Gamma_0(1)$ (see e.g. [23, Chap. 7]).

We shall denote by $\widetilde{M}_k^{\leq s}[\Gamma_0(N)]$ the space of quasimodular forms of weight k, depth $\leq s$ on $\Gamma_0(N)$ and $M_k[\Gamma_0(N)] = \widetilde{M}_k^{\leq 0}[\Gamma_0(N)]$ the space of modular forms of weight k on $\Gamma_0(N)$.

Our method for Theorem 1.1 is to remark that the function

$$\begin{aligned} H_N(z) &= E_2(z)E_2(Nz) \\ &= 1 - 24 \sum_{n=1}^{+\infty} \left[\sigma_1(n) + \sigma_1\left(\frac{n}{N}\right)\right] e^{2\pi i n z} + 576 \sum_{n=1}^{+\infty} W_N(n) e^{2\pi i n z} \end{aligned}$$

is a quasimodular form of weight 4, depth 2 on $\Gamma_0(N)$ that we linearize using the following lemma.

Lemma 1.17. *Let $k \geq 2$ even. Then*

$$\widetilde{M}_k^{\leq k/2}[\Gamma_0(N)] = \bigoplus_{i=0}^{k/2-1} D^i M_{k-2i}[\Gamma_0(N)] \oplus \mathbb{C} D^{k/2-1} E_2.$$

We have set

$$D := \frac{1}{2i\pi} \frac{\mathrm{d}}{\mathrm{d}z}.$$

Let $\{B_k\}_{k\in\mathbb{N}}$ be the sequence of rational numbers defined by its exponential generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{+\infty} B_k \frac{t^k}{k!}.$$

We shall use the Eisenstein series to express the basis we need:

$$E_{k,N}(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) e^{2\pi i n N z} \in M_k[\Gamma_0(N)]$$

for all $k \in 2\mathbb{N}^* + 2$, $N \in \mathbb{N}^*$. If $N = 1$ we simplify by writing $E_k := E_{k,N}$. For weight 2 forms, we shall need

$$\Phi_{a,b}(z) = \frac{1}{b-a}[bE_2(bz) - aE_2(az)] \in M_2[\Gamma_0(b)]$$

for all $b > 1$ and $a \mid b$.

Let χ be a Dirichlet character. If f satisfies all of what is needed to be a quasi-modular form except (1.3) being replaced by

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \chi(d) \sum_{i=0}^{n} f_i(z) \left(\frac{c}{cz+d}\right)^i,$$

then one says that f is a quasimodular form of weight k, depth s and character χ on $\Gamma_0(N)$. We denote by $\widetilde{M}_k^{\leq s}[\Gamma_0(N), \chi]$ the vector space of quasimodular forms of weight k, depth $\leq s$ and character χ on $\Gamma_0(N)$. If $\chi = \chi_0$ is a principal character of modulus dividing N, then $\widetilde{M}_k^{\leq s}[\Gamma_0(N), \chi] = \widetilde{M}_k^{\leq s}[\Gamma_0(N)]$.

If $f \in \widetilde{M}_k^{\leq s}[\Gamma_0(N)]$, then f has a Fourier expansion with coefficients $\{\widehat{f}(n)\}_{n \in \mathbb{N}}$. We define the twist of f by the Dirichlet character χ as

$$f \otimes \chi(z) = \sum_{n=0}^{+\infty} \chi(n) \widehat{f}(n) e^{2\pi i n z}.$$

In [17, Proposition 9], we proved the following proposition:

Proposition 1.18. *Let χ be a primitive Dirichlet character of conductor m. Let f be a quasimodular form of weight k and depth s on $\Gamma_0(N)$. Then $f \otimes \chi$ is a quasimodular form of weight k, depth less than or equal to s and character χ^2 on $\Gamma_0(\mathrm{lcm}(N, m^2))$.*

Remark 1.19. The condition of primitivity of the character may be replaced by the condition of non vanishing of its Gauss sum.

The proof of Theorem 1.9 follows from the linearization of $E_2 \cdot E_2 \otimes \chi_3$.

Theorems 1.10 and 1.11 follow from the linearization of derivatives of forms of type $E_j E_{k,N}$.

1.3. *Generalization of the results*

For $N \geq 1$ and $k \geq 2$, let $A^*_{N,k}$ be the set of triples (ψ, ϕ, t) such that ψ is a primitive Dirichlet character of modulus L, ϕ is a primitive Dirichlet character of modulus M and t is an integer such that $tLM \mid N$ (and $tLM \neq 1$ if $k = 2$) with the extra condition

$$\psi\phi(n) = \begin{cases} 1 & \text{if } (n, N) = 1 \\ 0 & \text{otherwise} \end{cases} \qquad (n \in \mathbb{N}^*). \tag{1.4}$$

We write **1** for the primitive character of modulus 1 (the constant function $n \mapsto 1$). We extend the definition of σ_k: for k and n in $\mathbb{N}^*$ we set

$$\sigma_k^{\psi,\phi}(n) := \sum_{d|n} \psi\left(\frac{n}{d}\right)\phi(d)d^k$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}^*$ we set $\sigma_k^{\psi,\phi}(n) = 0$. If M is the modulus of the primitive character ϕ, we define the sequence $\{B_k^{\phi}\}_{k\in\mathbb{N}}$ by its exponential generating function

$$\sum_{c=0}^{M-1} \phi(c)\frac{te^{ct}}{e^{Mt}-1} = \sum_{k=0}^{+\infty} B_k^{\phi}\frac{t^k}{k!}.$$

For any $(\psi, \phi, t) \in A_{N,k}^*$, define

$$E_k^{\psi,\phi}(z) := \delta(\psi = \mathbf{1}) - \frac{2k}{B_k^{\phi}}\sum_{n=1}^{+\infty} \sigma_{k-1}^{\psi,\phi}(n)e^{2\pi inz}$$

and

$$E_{k,t}^{\psi,\phi}(z) := \begin{cases} E_k^{\psi,\phi}(tz) & \text{if } (k,\psi,\phi) \neq (2,\mathbf{1},\mathbf{1}) \\ E_2^{\mathbf{1},\mathbf{1}}(z) - tE_2^{\mathbf{1},\mathbf{1}}(tz) & \text{otherwise} \end{cases}$$

where $\delta(\psi = \mathbf{1})$ is 1 if $\psi = \mathbf{1}$ and 0 otherwise.

For $N \geq 1$ and $k \geq 2$ even, the set

$$\{E_{k,t}^{\psi,\phi} \colon (\psi, \phi, t) \in A_{N,k}^*\}$$

is a basis for the orthogonal subspace (called Eisenstein subspace, the scalar product being the Petersson one) of the cuspidal subspace $S_k[\Gamma_0(N)]$ of $M_k[\Gamma_0(N)]$ [10, Chap. 4].

Moreover, by Atkin–Lehner–Li theory [10, Chap. 5], a basis for $S_k[\Gamma_0(N)]$ is

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^* \\ dM|N}} \alpha_d(H_k^*[\Gamma_0(M)])$$

where α_d is

$$\begin{aligned} \alpha_d \colon M_k[\Gamma_0(M)] &\to M_k[\Gamma_0(M)] \\ f &\mapsto [z \mapsto f(dz)] \end{aligned}$$

and $H_k^*[\Gamma_0(M)]$ is the set of primitive forms of weight k on $\Gamma_0(M)$.

A corollary is the following generalisation of Theorems 1.1 and 1.10. If f is a modular form, we denote by $\{\widehat{f}(n)\}_{n\in\mathbb{N}}$ the sequence of its Fourier coefficients.

Proposition 1.20. *Let $N \geq 1$. There exist scalars $a_{\psi,\phi,t}$, $a_{M,d,f}$ and a such that, for all $n \geq 1$, we have*

$$W_N(n) = \sum_{(\psi,\phi,t)\in A^*_{N,4}} a_{\psi,\phi,t}\sigma_3^{\psi,\phi}\left(\frac{n}{t}\right) + \sum_{(\psi,\phi,t)\in A^*_{N,2}} a_{\psi,\phi,t} n\sigma_1^{\psi,\phi}\left(\frac{n}{t}\right) + an\sigma_1(n)$$
$$+ \sum_{\substack{(d,M)\in\mathbb{N}^* \\ dM|N}} \sum_{f\in H_4^*[\Gamma_0(M)]} a_{M,d,f}\widehat{f}\left(\frac{n}{d}\right) + \sum_{\substack{(d,M)\in\mathbb{N}^* \\ dM|N}} \sum_{f\in H_2^*[\Gamma_0(M)]} a_{M,d,f} n\widehat{f}\left(\frac{n}{d}\right)$$
$$+ \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{N}\right).$$

More generally, for any $N \geq 1$ and any even $\ell \geq 4$, the arithmetic functions

$$n \mapsto \sum_{k<n/N} \sigma_1(n-kN)\sigma_{\ell-1}(k) - \frac{B_\ell}{2\ell}\sigma_1(n) - \frac{1}{24}\sigma_{\ell-1}\left(\frac{n}{N}\right)$$

and

$$n \mapsto \sum_{k<n/N} \sigma_1(k)\sigma_{\ell-1}(n-kN) - \frac{B_\ell}{2\ell}\sigma_1\left(\frac{n}{N}\right) - \frac{1}{24}\sigma_{\ell-1}(n)$$

are linear combinations of the sets of functions

$$\bigcup_{(\psi,\phi,t)\in A^*_{N,\ell+2}} \left\{n \mapsto \sigma_{\ell+1}^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{(\psi,\phi,t)\in A^*_{N,\ell}} \left\{n \mapsto n\sigma_{\ell-1}^{\psi,\phi}\left(\frac{n}{t}\right)\right\},$$

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^* \\ dM|N}} \bigcup_{f\in H^*_{\ell+2}[\Gamma_0(M)]} \left\{n \mapsto \widehat{f}\left(\frac{n}{d}\right)\right\},$$

$$\bigcup_{\substack{(d,M)\in\mathbb{N}^* \\ dM|N}} \bigcup_{f\in H^*_{\ell}[\Gamma_0(M)]} \left\{n \mapsto n\widehat{f}\left(\frac{n}{d}\right)\right\}.$$

The same allows to generalize Theorem 1.9. If $b \geq 1$ is an integer, denote by $X(b)$ the set of Dirichlet characters of modulus b. By orthogonality, we have

$$S[a,b](n) = \frac{1}{\varphi(b)} \sum_{\chi\in X(b)} \overline{\chi(a)} \sum_{m=1}^{n-1} \chi(m)\sigma_1(m)\sigma_1(n-m).$$

It follows that the function to be considered is now

$$\frac{1}{\varphi(b)} \sum_{\chi\in X(b)} \overline{\chi(a)} E_2 \cdot E_2 \otimes \chi.$$

We restrict to b squarefree so that the Gauss sum associates to any character of modulus b is non vanishing. For $N \geq 1$, let $\chi_N^{(0)}$ be the principal character of modulus N. For $\chi \in X(b)$, we define $A^*_{N,k,\chi}$ as $A^*_{N,k}$ except we replace condition (1.4) by

$$\psi\phi = \chi_N^{(0)}\chi.$$

Then, similarly to the Proposition 1.20, we have the following proposition:

Proposition 1.21. *Let $b \geq 1$ squarefree and $a \in [0, b-1]$ be integers. Then the function*

$$n \mapsto S[a,b](n) - \frac{1}{24}[\delta(b \mid a) + \delta(b \mid n-a)]\sigma_1(n)$$

is a linear combination of the set of functions

$$\bigcup_{\chi \in X(b)} \bigcup_{(\psi,\phi,t) \in A^*_{N,4,\chi}} \left\{ n \mapsto \sigma_3^{\psi,\phi}\left(\frac{n}{t}\right) \right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{(\psi,\phi,t) \in A^*_{N,2,\chi}} \left\{ n \mapsto n\sigma_1^{\psi,\phi}\left(\frac{n}{t}\right) \right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{\substack{(d,M) \in \mathbb{N}^* \\ dM \mid N}} \bigcup_{f \in H_4^*[\Gamma_0(M), \chi_N^{(0)}\chi]} \left\{ n \mapsto \widehat{f}\left(\frac{n}{d}\right) \right\},$$

$$\bigcup_{\chi \in X(b)} \bigcup_{\substack{(d,M) \in \mathbb{N}^* \\ dM \mid N}} \bigcup_{f \in H_2^*[\Gamma_0(M), \chi_N^{(0)}\chi]} \left\{ n \mapsto n\widehat{f}\left(\frac{n}{d}\right) \right\},$$

$$\{n \mapsto n\sigma_1(n)\}$$

where N is the least common multiple of 2 *and* b^2 *and* $\delta(b \mid n-a)$ *is* 1 *if* $n \equiv a (mod\ b)$ *and* 0 *otherwise.*

2. Convolution of Levels 3, 5, 6, 7, 8, 9, 10 and 11

2.1. *Level* 3

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(3)] = M_4[\Gamma_0(3)] \oplus DM_2[\Gamma_0(3)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(3)]$ has dimension 2 and is spanned by the two linearly independent forms E_4 and $E_{4,3}$. The vector space $M_2[\Gamma_0(3)]$ has dimension 1 and is spanned by $\Phi_{1,3}$. Computing the first Fourier coefficients, we therefore find that

$$H_3 = \frac{1}{10}E_4 + \frac{9}{10}E_{4,3} + 4D\Phi_{1,3} + 4DE_2. \tag{2.1}$$

Comparing with the Fourier expansion in (2.1) leads to the corresponding result in Theorem 1.1.

2.2. *Level* 5

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(5)] = M_4[\Gamma_0(5)] \oplus DM_2[\Gamma_0(5)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(5)]$ has dimension 3 and is spanned by the linearly independent forms E_4, $E_{4,5}$ and $\Delta_{4,5}$. The vector space $M_2[\Gamma_0(5)]$ has dimension 1 and is spanned by $\Phi_{1,5}$. Computing the first Fourier coefficients, we therefore find that

$$H_5 = \frac{1}{26}E_4 + \frac{25}{26}E_{4,5} - \frac{288}{65}\Delta_{4,5} + \frac{24}{5}D\Phi_{1,5} + \frac{12}{5}DE_2. \tag{2.2}$$

Comparing with the Fourier expansion in (2.2) leads to the corresponding result in Theorem 1.1.

2.3. *Level* 6

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(6)] = M_4[\Gamma_0(6)] \oplus DM_2[\Gamma_0(6)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(6)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_{4,2}$,$E_{4,3}$ $E_{4,6}$ and $\Delta_{4,6}$. The vector space $M_2[\Gamma_0(6)]$ has dimension 3 and is spanned by the three linearly independent forms $\Phi_{1,2}$, $\Phi_{1,3}$ and $\Phi_{3,6}$. Computing the first Fourier coefficients, we therefore find that

$$H_6 = \frac{1}{50}E_4 + \frac{2}{25}E_{4,2} + \frac{9}{50}E_{4,3} + \frac{18}{25}E_{4,6} - \frac{24}{5}\Delta_{4,6} + 2D\Phi_{1,3} + 3D\Phi_{3,6} + 2DE_2. \tag{2.3}$$

Comparing with the Fourier expansion in (2.3) leads to the corresponding result in Theorem 1.1.

2.4. *Level* 7

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(7)] = M_4[\Gamma_0(7)] \oplus DM_2[\Gamma_0(7)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(7)]$ has dimension 3 and is spanned by the three linearly independent forms E_4, $E_{4,7}$ and $\Delta_{4,7}$. The vector space $M_2[\Gamma_0(7)]$ has dimension 1 and is spanned by the form $\Phi_{1,7}$. Computing the first Fourier coefficients, we therefore find that

$$H_7 = \frac{1}{50}E_4 + \frac{49}{50}E_{4,7} - \frac{288}{35}\Delta_{4,7} + \frac{36}{7}D\Phi_{1,7} + \frac{12}{7}DE_2. \tag{2.4}$$

Comparing with the Fourier expansion in (2.4) leads to the corresponding result in Theorem 1.1.

2.5. *Level* 8

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(8)] = M_4[\Gamma_0(8)] \oplus DM_2[\Gamma_0(8)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(8)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_{4,2}$, $E_{4,4}$, $E_{4,8}$ and $\Delta_{4,8}$. The vector space $M_2[\Gamma_0(8)]$ has dimension 3 and is spanned by the forms $\Phi_{1,4}$, $\Phi_{1,8}$ and

$$\Phi_{1,4,2} := z \mapsto \Phi_{1,4}(2z).$$

Computing the first Fourier coefficients, we therefore find that

$$H_8 = \frac{1}{80}E_4 + \frac{3}{80}E_{4,2} + \frac{3}{20}E_{4,4} + \frac{4}{5}E_{4,8} - 9\Delta_{4,8} + \frac{21}{4}D\Phi_{1,8} + \frac{3}{2}DE_2. \tag{2.5}$$

Comparing with the Fourier expansion in (2.5) leads to the corresponding result in Theorem 1.1.

2.6. *Level* 9

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(9)] = M_4[\Gamma_0(9)] \oplus DM_2[\Gamma_0(9)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(9)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_4 \otimes \chi_3$, $E_{4,3}$, $E_{4,9}$ and $\Delta_{4,9}$. The vector space $M_2[\Gamma_0(9)]$ has dimension 3 and is spanned by the forms $\Phi_{1,3}$, $\Phi_{1,3} \otimes \chi_3$ and $\Phi_{1,9}$.

Computing the first Fourier coefficients, we therefore find that

$$H_9 = \frac{1}{90}E_4 + \frac{4}{45}E_{4,3} + \frac{9}{10}E_{4,9} - \frac{32}{3}\Delta_{4,9} + \frac{16}{3}D\Phi_{1,9} + \frac{4}{3}DE_2. \tag{2.6}$$

Comparing with the Fourier expansion in (2.6) leads to the corresponding result in Theorem 1.1.

2.7. *Level* 10

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(10)] = M_4[\Gamma_0(10)] \oplus DM_2[\Gamma_0(10)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(10)]$ has dimension 7 and is spanned by the seven linearly independent forms E_4, $E_{4,2}$, $E_{4,5}$, $E_{4,10}$, $\Delta_{4,10}$, $\Delta_{4,5}$ and

$$\Delta_{4,5,2} := z \mapsto \Delta_{4,5}(2z).$$

The vector space $M_2[\Gamma_0(10)]$ has dimension 3 and is spanned by the forms $\Phi_{1,10}$, $\Phi_{1,5}$ and

$$\Phi_{1,5,2} := z \mapsto \Phi_{1,5}(2z).$$

Computing the first Fourier coefficients, we therefore find that

$$H_{10} = \frac{1}{130}E_4 + \frac{2}{65}E_{4,2} + \frac{5}{26}E_{4,5} + \frac{10}{13}E_{4,10} - \frac{24}{5}\Delta_{4,10} - \frac{432}{65}\Delta_{4,5} - \frac{1728}{65}\Delta_{4,5,2} + \frac{27}{5}D\Phi_{1,10} + \frac{6}{5}DE_2. \tag{2.7}$$

Comparison with the Fourier expansion in (2.7) leads to the corresponding result in Theorem 1.1.

2.8. *Level* 11

By Lemma 1.17, we have

$$\widetilde{M_4^{\leq 2}}[\Gamma_0(11)] = M_4[\Gamma_0(11)] \oplus DM_2[\Gamma_0(11)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(11)]$ has dimension 4 and is spanned by the four linearly independent forms E_4, $E_{4,11}$, $\Delta_{4,11,1}$ and $\Delta_{4,11,2}$. Let F_1 be the parabolic form of weight 4 and level 11 given by

$$F_1(z) = [\Delta(z)\Delta(11z)]^{1/6} = e^{4\pi iz} - 4e^{6\pi iz} + 2e^{8\pi iz} + 8e^{10\pi iz} + O(e^{12\pi iz}).$$

Let T_2 be the Hecke operator given by

$$T_2 : \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imz} \mapsto \sum_{m\in\mathbb{Z}} \left[\sum_{\substack{d\in\mathbb{N} \\ d|(m,2) \\ (d,11)=1}} d^{k-1}\widehat{f}\left(\frac{2m}{d^2}\right)\right] e^{2\pi imz}.$$

It sends a parabolic form of weight 4 and level 11 to another one. Let

$$F_2 = T_2F_1 = e^{2\pi iz} + 2e^{4\pi iz} - 5e^{6\pi iz} - 2e^{8\pi iz} + 9e^{10\pi iz} + O(e^{12\pi iz}).$$

There exists λ_1 and λ_2 such that

$$\Delta_{4,11,1} = F_2 + \lambda_1 F_1 \quad \text{and} \quad \Delta_{4,11,2} = F_2 + \lambda_2 F_1.$$

For $j \in \{1,2\}$, it follows that

$$\tau_{4,11,j}(2) = 2 + \lambda_j \quad \text{and} \quad \tau_{4,11,j}(4) = -2 + 2\lambda_j.$$

Since $\Delta_{4,11,j}$ is primitive, it satisfies (1.2) hence $\lambda_j^2 - 2\lambda_j - 2 = 0$. In other words

$$X^2 - 2X - 2 = (X - \lambda_1)(X - \lambda_2).$$

This provides a way to compute the Fourier coefficients of $\Delta_{4,11,1}$ and $\Delta_{4,11,2}$ from the ones of Δ and proves that these coefficients live in $\mathbb{Q}(t)$ where t is a root of $X^2 - 2X - 2$.

The vector space $M_2[\Gamma_0(11)]$ has dimension 2 and is spanned by the form $\Phi_{1,11}$ and its unique primitive form

$$\Delta_{2,11} = [\Delta(z)\Delta(11z)]^{1/12}.$$

Table 14. First Fourier coefficients of $\Delta_{2,11}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{2,11}(n)$	1	−2	−1	2	1	2	−2	0	−2	−2	1
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{2,11}(n)$	−2	4	4	−1	−4	−2	4	0	2	2	−2

Table 15. First Fourier coefficients of $\Delta_{4,13,1}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{4,13,1}(n)$	1	−5	−7	17	−7	35	−13	−45	22	35	−26
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{4,13,1}(n)$	−119	13	65	49	89	77	−110	−126	−119	91	130

Computing the first Fourier coefficients, we therefore find that

$$H_{11} = \frac{1}{122}E_4 + \frac{121}{122}E_{4,11} - \frac{192t+4128}{671}\Delta_{4,11,1} + \frac{192t-4512}{671}\Delta_{4,11,2} + \frac{60}{11}D\Phi_{1,11} + \frac{12}{11}DE_2. \tag{2.8}$$

Comparison with the Fourier expansion in (2.8) leads to Theorem 1.3.

2.9. *Level* 13

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(13)] = M_4[\Gamma_0(13)] \oplus DM_2[\Gamma_0(13)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(13)]$ has dimension 5 and is spanned by the five linearly independent forms E_4, $E_{4,13}$, $\Delta_{4,13,1}$, $\Delta_{4,13,2}$ and $\Delta_{4,13,3}$. The vector space $M_2[\Gamma_0(13)]$ has dimension 1 and is spanned by the form $\Phi_{1,13}$.

Computing the first Fourier coefficients, we therefore find that

$$H_{13} = \frac{1}{170}E_4 + \frac{169}{170}E_{4,13} + \frac{288u-1728}{221}\Delta_{4,13,2} - \frac{288u+1440}{221}\Delta_{4,13,3} + \frac{70}{13}D\Phi_{1,13} + \frac{12}{13}DE_2. \tag{2.9}$$

Comparison with the Fourier expansion in (2.9) leads to the Theorem 1.5.

2.10. *Level* 14

By Lemma 1.17, we have

$$\widetilde{M}_4^{\leq 2}[\Gamma_0(14)] = M_4[\Gamma_0(14)] \oplus DM_2[\Gamma_0(14)] \oplus \mathbb{C}DE_2.$$

The vector space $M_4[\Gamma_0(14)]$ has dimension 8 and is spanned by the eight linearly independent forms E_4, $E_{4,2}$, $E_{4,7}$, $E_{4,14}$, $\Delta_{4,7}$,

$$F_{4,7,2} \colon z \mapsto \Delta_{4,7}(2z),$$

and the two primitive forms $\Delta_{4,14,1}$ and $\Delta_{4,14,2}$. Another basis of the subspace of parabolic forms is $\Delta_{4,7}$, $F_{4,7,2}$, $\Delta_{2,14}^2$ and $\Delta_{2,14}\Phi_{1,14}$ where

$$\Delta_{2,14}(z) := [\Delta(z)\Delta(2z)\Delta(7z)\Delta(14z)]^{1/24}$$

is the unique primitive form of weight 2 on $\Gamma_0(14)$. We echelonize this second basis by defining

$$J_1 = -\frac{11}{28}\Delta_{4,7} - \frac{22}{7}F_{4,7,2} + \frac{11}{7}\Delta_{2,14}^2 + \frac{39}{8}\Delta_{2,4}\Phi_{1,14} = e^{2\pi iz} + O\left(e^{10\pi iz}\right),$$

$$J_2 = -\frac{13}{56}\Delta_{4,7} + \frac{1}{7}F_{4,7,2} + \frac{3}{7}\Delta_{2,14}^2 + \frac{13}{56}\Delta_{2,4}\Phi_{1,14} = e^{4\pi iz} + O\left(e^{10\pi iz}\right),$$

$$J_3 = \frac{13}{56}\Delta_{4,7} + \frac{19}{14}F_{4,7,2} - \frac{13}{14}\Delta_{2,14}^2 - \frac{13}{56}\Delta_{2,4}\Phi_{1,14} = e^{6\pi iz} + O\left(e^{10\pi iz}\right),$$

$$J_4 = -\frac{13}{56}\Delta_{4,7} - \frac{6}{7}F_{4,7,2} + \frac{3}{7}\Delta_{2,14}^2 + \frac{13}{56}\Delta_{2,4}\Phi_{1,14} = e^{8\pi iz} + O\left(e^{10\pi iz}\right).$$

We then have

$$\Delta_{4,14,j} = J_1 + b_jJ_2 + c_jJ_3 + d_jJ_4.$$

From $\tau_{4,14,j}(4) = \tau_{4,14,j}(2)^2$ we deduce $d_j = b_j^2$. Then, from $\tau_{4,14,j}(6) = \tau_{4,14,j}(2)$ $\tau_{4,14,j}(3)$ and $\tau_{4,14,j}(8) = \tau_{4,14,j}(2)\tau_{4,14,j}(4)$ we respectively deduce

$$2b_j + b_jc_j + 2c_j = -4,$$
$$b_j^3 - b_j^2 + 6b_j + 4c_j = 8$$

i.e.

$$c_j = -\frac{1}{4}b_j^3 + \frac{1}{4}b_j^2 - \frac{3}{2}b_j + 2$$

and

$$(b_j - 2)(b_j + 2)(b_j^2 + b_j + 8) = 0.$$

Since the coefficients of $\Delta_{4,14,j}$ are all totally real, we must have

$$\Delta_{4,14,1} = J_1 + 2J_2 - 2J_3 + 4J_4,$$
$$\Delta_{4,14,2} = J_1 - 2J_2 + 8J_3 + 4J_4.$$

Finally,

$$\Delta_{4,14,1} = -\frac{9}{4}\Delta_{4,7} - 9F_{4,7,2} + 6\Delta_{2,14}^2 + \frac{13}{4}\Delta_{2,14}\Phi_{1,14}, \tag{2.10}$$

$$\Delta_{4,14,2} = \Delta_{4,7} + 4F_{4,7,2} - 5\Delta_{2,14}^2. \tag{2.11}$$

Equations (2.10) and (2.11) allow to compute the first terms of the sequences $\tau_{4,14,1}$ and $\tau_{4,14,2}$. The vector space $M_2[\Gamma_0(14)]$ has dimension 4 and is spanned by the forms $\Phi_{1,7}$, $\Phi_{1,14}$, $\Phi_{2,14}$ and its unique primitive form $\Delta_{2,14}$.

Table 16. First Fourier coefficients of $\Delta_{2,14}$.

n	1	2	3	4	5	6	7	8	9	10	11
$\tau_{2,14}(n)$	1	−1	−2	1	0	2	1	−1	1	0	0
n	12	13	14	15	16	17	18	19	20	21	22
$\tau_{2,14}(n)$	−2	−4	−1	0	1	6	−1	2	0	−2	0

Computing the first Fourier coefficients, we therefore find that

$$\begin{aligned} H_{14} = {} & \frac{1}{250}E_4 + \frac{2}{125}E_{4,2} + \frac{49}{250}E_{4,7} + \frac{98}{125}E_{4,14} - \frac{864}{175}\Delta_{4,7} - \frac{3456}{175}F_{4,7,2} \\ & - \frac{48}{7}\Delta_{4,14,1} - \frac{72}{25}\Delta_{4,14,2} + \frac{39}{7}D\Phi_{1,14} + \frac{6}{7}DE_2. \end{aligned} \tag{2.12}$$

Comparison with the Fourier expansion in (2.12) leads to Theorem 1.7.

3. Convolutions of Level 1, 2, 4

The convolutions of level dividing 4 were evaluated in [17, Proposition 7]. We obtained

$$W_1(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n)$$

from the equality

$$E_2^2 = E_4 + 12DE_2$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(1)]$;

$$W_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right)$$

from the equality

$$H_2 = \frac{1}{5}E_4 + \frac{4}{5}E_{4,2} + 3D\Phi_{1,2} + 6DE_2$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(2)]$; and

$$\begin{aligned} W_4(n) = {} & \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right) \\ & + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right) \end{aligned}$$

from the equality

$$H_4 = \frac{1}{20}E_4 + \frac{3}{20}E_{4,2} + \frac{4}{5}E_{4,4} + \frac{9}{2}D\Phi_{1,4} + 3DE_2$$

in $\widetilde{M}_4^{\leq 2}[\Gamma_0(4)]$.

4. Twisted Convolution Sums

Let $\chi_3^{(0)}$ be the principal character of modulus 3. Remarking that

$$S[0,3](n) = \sum_{a+b=n} \sigma_1(a)\sigma_1(b) - \sum_{a+b=n} \chi_3^{(0)}(a)\sigma_1(a)\sigma_1(b),$$

we consider $E_2^2 - E_2(E_2 \otimes \chi_3^{(0)})$. Since $E_2 \otimes \chi_3^{(0)} \in \widetilde{M}_2^{\leq 1}[\Gamma_0(9), \chi_3^{(0)}] = \widetilde{M}_2^{\leq 1}[\Gamma_0(9)]$, we have

$$E_2^2 - E_2(E_2 \otimes \chi_3^{(0)}) \in \widetilde{M}_4^{\leq 2}[\Gamma_0(9)].$$

We use the same method and notations as in Sec. 2.6. We compute

$$\begin{aligned} E_2^2 - E_2(E_2 \otimes \chi_3^{(0)}) = {} & \frac{11}{30}E_4 + \frac{10}{3}E_{4,3} - \frac{27}{10}E_{4,9} + 32\Delta_{4,9} \\ & + 16D\Phi_{1,3} - 16D\Phi_{1,9} + 12DE_2. \end{aligned}$$

The evaluation of $S[1,3]$ given in Theorem 1.9 follows by comparison of the Fourier expansions.

We compute $S[1,3]$ after having remarked that

$$\frac{\chi_3^{(0)}(n) + \chi_3(n)}{2} = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the function to be linearized here is

$$\frac{1}{2}E_2[E_2 \otimes \chi_3^{(0)} + E_2 \otimes \chi_3]$$

whose nth Fourier coefficient ($n \in \mathbb{N}^*$) is

$$-24\delta(3 \mid n-1)\sigma_1(n) + 576S[1,3](n).$$

This is again a quasimodular form in $\widetilde{M}_4^{\leq 2}[\Gamma_0(9)]$, and as in Sec. 2.6, we linearize it as

$$\frac{19}{60}E_4 + \frac{1}{20}E_4 \otimes \chi_3 - \frac{5}{3}E_{4,3} + \frac{27}{20}E_{4,9} + 32\Delta_{4,9} - 8D\Phi_{1,3} - 6D(\Phi_{1,3} \otimes \chi_3) + 8D\Phi_{1,9}.$$

The evaluation of $S[1,3]$ given in Theorem 1.9 follows by comparison of the Fourier expansions.

The evaluation of $S[2,3]$ follows immediately from

$$S[0,3](n) + S[1,3](n) + S[2,3](n) = W_1(n)$$

and Theorem 1.1.

5. On Identities by Melfi

The first three identities of Theorem 1.10 are a direct consequence of the following ones:

$$E_2E_4 \in \widetilde{M}_6^{\leq 1}[\Gamma_0(1)] = M_6[\Gamma_0(1)] \oplus DM_4[\Gamma_0(1)] = \mathbb{C}E_6 \oplus \mathbb{C}DE_4,$$

and

$$\begin{aligned} E_2E_{4,2},\ E_4E_{2,2} \in \widetilde{M}_6^{\leq 1}[\Gamma_0(2)] &= M_6[\Gamma_0(2)] \oplus DM_4[\Gamma_0(2)] \\ &= \mathbb{C}E_6 \oplus \mathbb{C}E_{6,2} \oplus \mathbb{C}DE_4 \oplus \mathbb{C}DE_{4,2} \end{aligned}$$

which imply by comparison of the first Fourier coefficients

$$E_2E_4 = E_6 + 3DE_4,$$

$$E_2E_{4,2} = \frac{1}{21}E_6 + \frac{20}{21}E_{6,2} + 3DE_{4,2}$$

and

$$E_4E_{2,2} = \frac{5}{21}E_6 + \frac{16}{21}E_{6,2} + \frac{3}{2}DE_4.$$

The last three identities of Theorem 1.10 are a direct consequence of the following ones:

$$E_2E_6 \in \widetilde{M}_8^{\leq 1}[\Gamma_0(1)] = M_8[\Gamma_0(1)] \oplus DM_6[\Gamma_0(1)] = \mathbb{C}E_8 \oplus \mathbb{C}DE_6,$$

and

$$\begin{aligned} E_{2,2}E_6,\ E_2E_{6,2} \in \widetilde{M}_8^{\leq 1}[\Gamma_0(2)] &= M_8[\Gamma_0(2)] \oplus DM_6[\Gamma_0(2)] \\ &= \mathbb{C}E_8 \oplus \mathbb{C}E_{8,2} \oplus \mathbb{C}\Delta_{8,2} \oplus \mathbb{C}DE_6 \oplus \mathbb{C}DE_{6,2} \end{aligned}$$

which imply by comparison of the first Fourier coefficients

$$E_2E_6 = E_8 + 2DE_6,$$

$$E_{2,2}E_6 = \frac{21}{85}E_8 + \frac{64}{85}E_{8,2} - \frac{2016}{17}\Delta_{8,2} + DE_6$$

and

$$E_2E_{6,2} = \frac{1}{85}E_8 + \frac{84}{85}E_{8,2} - \frac{504}{17}\Delta_{8,2} + 2DE_{6,2}.$$

6. On Some Identities of Lahiri Type

6.1. *Method*

For $\mathbf{a} := (a_1, \dots, a_r) \in \mathbb{N}^r$, $\mathbf{b} := (b_1, \dots, b_r) \in (2\mathbb{N}+1)^r$ and $\mathbf{N} := (N_1, \dots, N_r) \in \mathbb{N}^{*r}$ the sum $S[\mathbf{a}, \mathbf{b}, \mathbf{N}]$ defined in (1.1) is relied to the quasimodular forms via the function

$$D^{a_1} E_{b_1+1,N_1} \cdots D^{a_r} E_{b_r+1,N_r} \in \widetilde{M}^{\leq a_1+\cdots+a_r+t(\mathbf{b})}_{b_1+\cdots+b_r+r+2(a_1+\cdots+a_r)}[\Gamma_0(\operatorname{lcm}(N_1, \dots, N_r))] \tag{6.1}$$

where

$$t(\mathbf{b}) = \#\{i \in \{1, \dots, r\} : b_i = 1\}.$$

Since we always can consider that the coordinates of $\mathbf{a}$ are given in increasing order, let ℓ be the nonnegative integer such that $a_1 = \cdots = a_\ell = 0$ and $a_{\ell+1} \neq 0$ (we take $\ell = 0$ if $\mathbf{a}$ has all its coordinates positive). We consider the function

$$\begin{aligned}\Psi_{\mathbf{a},\mathbf{b},\mathbf{N}} &:= \prod_{j=1}^{\ell} (E_{b_j+1,N_1} - 1) \prod_{j=\ell+1}^{r} D^{a_j} E_{b_j+1,N_j} \\ &\in \bigoplus_{k=b_{j+1}+\cdots+b_r+r+2(a_{j+1}+\cdots+a_r)-j}^{b_1+\cdots+b_r+r+2(a_{j+1}+\cdots+a_r)} \widetilde{M}^{\leq a_1+\cdots+a_r+t(\mathbf{b})}_{k}[\Gamma_0(\operatorname{lcm}(N_1, \dots, N_r))].\end{aligned}$$

We have

$$\Psi_{\mathbf{a},\mathbf{b},\mathbf{N}}(z) = (-2)^r \left[\prod_{j=1}^{r} \frac{b_j+1}{B_{b_j+1}}\right] \sum_{n=1}^{+\infty} S[\mathbf{a}, \mathbf{b}, \mathbf{N}] e^{2\pi i n z}.$$

The evaluation of $S[(0,1,1),(1,1,1)]$ is a consequence (by Lemma 1.17) of

$$\begin{aligned}(E_1 - 1)(DE_2)^2 \in \mathbb{C}E_8 \oplus \mathbb{C}DE_6 \oplus \mathbb{C}D^2E_4 \oplus \mathbb{C}D^3E_2 \oplus \mathbb{C}E_{10} \\ \oplus \mathbb{C}DE_8 \oplus \mathbb{C}D^2E_6 \oplus \mathbb{C}D^3E_4 \oplus \mathbb{C}D^4E_2.\end{aligned}$$

The comparison of the first Fourier coefficients leads to

$$(E_2 - 1)(DE_2)^2 = -\frac{1}{5}D^2E_4 - 2D^3E_2 + \frac{2}{21}D^2E_6 + \frac{4}{5}D^3E_4 + 6D^4E_2.$$

Hence the evaluation of $S[(0,1,1),(1,1,1)]$ given in Theorem 1.11.

The evaluation of $S[(0,0,0,1,1),(1,1,1,1,1)]$ is a consequence (by Lemma 1.17) of

$$(E_2 - 1)^3(DE_2)^2 \in \mathbb{C}\Delta \oplus \mathbb{C}D\Delta \bigoplus_{i=1}^{7} \bigoplus_{j=0}^{7-i} \mathbb{C}D^j E_{2i}.$$

The comparison of the first Fourier coefficients leads to

$$
\begin{aligned}
(E_2-1)^3(DE_2)^2 = &-\frac{8}{35}\Delta + \frac{8}{35}D\Delta - 2D^3E_2 + 18D^4E_2 - \frac{216}{5}D^5E_2 + \frac{144}{5}D^6E_2 \\
&-\frac{1}{5}D^2E_4 + \frac{12}{5}D^3E_4 - \frac{234}{35}D^4E_4 + \frac{171}{35}D^5E_4 + \frac{2}{7}D^2E_6 \\
&-\frac{9}{7}D^3E_6 + \frac{25}{21}D^4E_6 - \frac{1}{6}D^2E_8 + \frac{4}{15}D^3E_8 + \frac{2}{55}D^2E_{10}.
\end{aligned}
$$

Hence the evaluation of $S[(0,0,0,1,1),(1,1,1,1,1)]$ given in Theorem 1.11.

We leave the proofs of Propositions 1.12 and 1.13 to the reader. They are obtained from the linearizations of

$$
(E_{2,2}-1)DE_{2,5} \in \widetilde{M}_4^{\leq 2}[\Gamma_0(5)] \oplus \widetilde{M}_6^{\leq 3}[\Gamma_0(10)] \tag{6.2}
$$

and

$$
DE_2DE_{2,5} \in \widetilde{M}_8^{\leq 5}[\Gamma_0(5)].
$$

6.2. *Primitive forms of weight* 6 *and level* 5 *or* 10

For the evaluation of (6.2) we remark that $\Delta_{4,5}\Phi_{1,5}$ is a parabolic modular form of weight 6 and level 5. Since the dimension of these forms is 1, we have

$$
\Delta_{6,5}(z) = \frac{1}{4}[\Delta(z)\Delta(5z)]^{1/6}[5E_2(5z) - E_2(z)]. \tag{6.3}
$$

Equation (6.3) provides a way to compute the few needed values of $\tau_{6,5}$.

We also give expressions for $\Delta_{6,10,i}$ where $i \in \{1,2,3\}$. We shall use the second Hecke operator of level 10 given by

$$
T_2 : \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imz} \mapsto \sum_{m\in\mathbb{Z}} \left[\sum_{\substack{d\in\mathbb{N} \\ d|(m,2) \\ (d,10)=1}} d^{k-1}\widehat{f}\left(\frac{2m}{d^2}\right) \right] e^{2\pi imz} = \sum_{m\in\mathbb{Z}} \widehat{f}(2m)e^{2\pi imz}.
$$

The space of parabolic modular forms of weight 6 and level 10 has dimension 5. A basis is given by

$$
\begin{aligned}
\Delta_{6,5}(z) &= [\Delta(z)\Delta(5z)]^{1/6}\Phi_{1,5}(z), \\
\Delta_{6,5,2}(z) &= \Delta_{6,5}(2z), \\
F(z) &= 3[\Delta(z)\Delta(5z)]^{1/6}\Phi_{1,10}(z), \\
F_1(z) &= [\Delta(z)\Delta(5z)]^{1/6}\Phi_{1,2}(z), \\
F_2(z) &= T_2F(z).
\end{aligned}
$$

To simplify the computations, we use an echelonized basis:

$$\begin{aligned}
V_1 &= -\frac{4}{15}\Delta_{6,5} + \frac{31}{10}\Delta_{6,5,2} + \frac{15}{32}F + \frac{1}{96}F_1 + \frac{3}{80}F_2 = e(z) + O(e(6z)),\\
V_2 &= \frac{1}{20}\Delta_{6,5} + \frac{6}{5}\Delta_{6,5,2} + \frac{1}{80}F_2 = e(2z) + O(e(6z)),\\
V_3 &= -\frac{1}{30}\Delta_{6,5} + \frac{7}{10}\Delta_{6,5,2} + \frac{1}{32}F - \frac{1}{96}F_1 + \frac{1}{80}F_2 = e(3z) + O(e(6z)),\\
V_4 &= -\frac{1}{40}\Delta_{6,5} - \frac{1}{10}\Delta_{6,5,2} - \frac{1}{160}F_2 = e(4z) + O(e(6z)),\\
V_5 &= \frac{1}{75}\Delta_{6,5} - \frac{11}{50}\Delta_{6,5,2} - \frac{11}{800}F - \frac{1}{480}F_1 - \frac{3}{400}F_2 = e(5z) + O(e(6z)).
\end{aligned}$$

We deduce

$$\Delta_{6,10,i} = V_1 + b_iV_2 + c_iV_3 + d_iV_4 + e_iV_5$$

since $\widehat{\Delta_{6,10,i}}(1) = 1$. Now, from $\widehat{\Delta_{6,10,i}}(4) = \widehat{\Delta_{6,10,i}}(2)^2$, we get $d_i = b_i^2$ so that

$$\Delta_{6,10,i} = V_1 + b_iV_2 + c_iV_3 + b_i^2V_4 + e_iV_5.$$

Next, from $\widehat{\Delta_{6,10,i}}(6) = \widehat{\Delta_{6,10,i}}(2)\widehat{\Delta_{6,10,i}}(3)$, and $\widehat{\Delta_{6,10,i}}(8) = \widehat{\Delta_{6,10,i}}(2)\widehat{\Delta_{6,10,i}}(4)$, we respectively get

$$2c_i + b_ic_i + 2e_i = 10 - 2b_i - b_i^2 \tag{6.4}$$

and

$$8c_i - 8e_i = 24 + 16b_i + 6b_i^2 + b_i^3. \tag{6.5}$$

Equations (6.4) and (6.5) give either $b_i = -4$ or $b_i \neq -4$ and

$$c_i = 4 - \frac{1}{2}b_i + \frac{1}{4}b_i^2$$

and

$$e_i = 1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3.$$

We first deal with the case $b_i \neq -4$. We then get

$$\Delta_{6,10,i} = V_1 + b_iV_2 + \left(4 - \frac{1}{2}b_i + \frac{1}{4}b_i^2\right)V_3 + b_i^2V_4 + \left(1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3\right)V_5.$$

Using $\widehat{\Delta_{6,10,i}}(10) = \widehat{\Delta_{6,10,i}}(2)\widehat{\Delta_{6,10,i}}(5)$, we obtain

$$\begin{aligned}
&b_i\left(1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3\right)\\
&\quad = -30 + 15b_i - 10\left(4 - \frac{1}{2}b_i + \frac{1}{4}b_i^2\right) + 5b_i^2 + 6\left(1 - \frac{5}{2}b_i - \frac{1}{2}b_i^2 - \frac{1}{8}b_i^3\right)
\end{aligned}$$

from what we get

$$b_i \in \{-4, 4, 1 - i\sqrt{31}, 1 + i\sqrt{31}\}.$$

The solution $b_i = -4$ is in this case not allowed whereas the solution $1 \pm i\sqrt{31}$ are not possible since the coefficients of a primitive form are totally real algebraic numbers [9]. We thus obtain a first primitive form:

$$\Delta_{6,10,1} = V_1 + 4V_2 + 6V_3 + 16V_4 - 25V_5.$$

We assume now that $b_i = -4$ so that

$$\Delta_{6,10,i} = V_1 - 4V_2 + c_iV_3 + 16V_4 + (c_i + 1)V_5.$$

From $\widehat{\Delta_{6,10,i}}(15) = \widehat{\Delta_{6,10,i}}(3)\widehat{\Delta_{6,10,i}}(5)$, we obtain $c_i = 24$ or $c_i = -26$. We hence get the two other primitive forms

$$\Delta_{6,10,2} = V_1 - 4V_2 + 24V_3 + 16V_4 + 25V_5$$

and

$$\Delta_{6,10,3} = V_1 - 4V_2 - 26V_3 + 16V_4 - 25V_5.$$

We deduce the following expressions:

$$\Delta_{6,10,1} = -\Delta_{6,5} + 16\Delta_{6,5,2} + F + \frac{1}{4}F_2, \tag{6.6}$$

$$\Delta_{6,10,2} = -\frac{4}{3}\Delta_{6,5} + 8\Delta_{6,5,2} + \frac{7}{8}F - \frac{7}{24}F_1, \tag{6.7}$$

$$\Delta_{6,10,3} = -\frac{1}{3}\Delta_{6,5} - 16\Delta_{6,5,2} + \frac{1}{3}F_1 - \frac{1}{4}F_2. \tag{6.8}$$

Equations (6.6)–(6.8) provide a way to compute the few needed values of $\tau_{6,10,i}$ for $i \in \{1, 2, 3\}$.

6.3. *Primitive forms of weight* 8 *and level* 5

The method is the same as in Sec. 6.2 so we will be more brief. The space of parabolic forms of weight 8 and level 5 has dimension 3 and a basis is

$$G_1(z) = [\Delta(z)\Delta(5z)]^{1/3},$$
$$G_2(z) = [\Delta(z)\Delta(5z)]^{1/6}\Phi_{1,5}(z)^2,$$
$$G_3 = -\frac{1}{24}[E_4, \Phi_{1,2}]_1,$$

where $[\ ,\ ]_1$ is the Rankin–Cohen bracket here defined by

$$[E_4, \Phi_{1,2}]_1 = \frac{1}{2i\pi}(4E_4\Phi'_{1,5} - 2E'_4\Phi_{1,5})$$

(see [29, Part 1, Sec. E] or [19, Part I, Sec. 6] for more details). We echelonize this basis by defining:

$$W_1 = \frac{46}{25}G_1 + \frac{82}{25}G_2 - \frac{3}{25}G_3 = e(z) + O(e(4z)),$$

$$W_2 = \frac{47}{375}G_1 - \frac{76}{375}G_2 + \frac{4}{375}G_3 = e(2z) + O(e(4z)),$$

$$W_3 = -\frac{41}{375}G_1 - \frac{19}{750}G_2 + \frac{1}{750}G_3 = e(3z) + O(e(4z)).$$

The primitive forms are then

$$\Delta_{8,5,i} = W_1 + b_i W_2 + c_i W_3.$$

From $\widehat{\Delta_{8,5,i}}(4) = \widehat{\Delta_{8,5,i}}(2)^2 - 2^7$ and $\widehat{\Delta_{8,5,i}}(6) = \widehat{\Delta_{8,5,i}}(2)\widehat{\Delta_{8,5,i}}(2)$ we get

$$c_i = 78 + 2b_i - \frac{1}{2}b_i^2,$$
$$(b_i + 14)(b_i^2 - 20b_i + 24) = 0.$$

Finally, defining t as one of the roots of $X^2 - 20X + 24$, we get

$$\Delta_{8,5,1} = \frac{16}{3}G_1 + \frac{22}{3}G_2 - \frac{1}{3}G_3, \tag{6.9}$$
$$\Delta_{8,5,2} = (12 - t)G_1 + G_2, \tag{6.10}$$
$$\Delta_{8,5,3} = (t - 8)G_1 + G_2. \tag{6.11}$$

Equations (6.9)–(6.11) provide a way to compute the few needed values of $\tau_{8,5,i}$ for $i \in \{1, 2, 3\}$.

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