Orbitwise Countings in $\mathcal{H}(2)$ and Quasimodular Forms

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We prove formulae for the countings by orbit of square-tiled surfaces of genus two with one singularity. These formulae were conjectured by Hubert and Lelièvre. We show that these countings admit quasimodular forms as generating functions.

1   Introduction

The main result of this paper is the proof of a conjecture of Hubert and Lelièvre.

Theorem 1.1. For odd n, the countings by orbit of primitive square-tiled surfaces of the stratum $\mathcal{H}(2)$ tiled with n squares are the following. Orbit A_n contains

$$a_n^p = \frac{3}{16}(n-1)n^2 \prod_{p|n}\left(1 - \frac{1}{p^2}\right) \tag{1.1}$$

primitive surfaces with n squares and orbit B_n contains

$$b_n^p = \frac{3}{16}(n-3)n^2 \prod_{p|n}\left(1 - \frac{1}{p^2}\right) \tag{1.2}$$

primitive surfaces with $n \geq 3$ squares and 0 for $n = 1$ square. $\qquad\square$

Remark 1.2. The notation $\prod_{p|n}$ indicates a product over prime divisors of n. The superscript p is here to emphasize primitivity.

Theorem 1.1 can also be expressed in terms of quasimodularity of the generating functions of the countings. More precisely, we have the following.

Corollary 1.3. For any odd positive integer n, the number a_n of n-square-tiled surfaces of type A in $\mathcal{H}(2)$, primitive or not, is the nth coefficient of the quasimodular form

$$\sum_{n=0}^{+\infty} a_n \exp(2i\pi nz) = \frac{1}{1280}\left[E_4(z) + 10\frac{d}{2i\pi\, dz}E_2(z) \right] \tag{1.3}$$

of weight 4 and depth 2 on $\mathrm{SL}(2,\mathbb{Z})$. $\qquad\square$

Remark 1.4. The functions E_2 and E_4 are the usual Eisenstein series of weights 2 and 4, respectively. They are defined in (4.2) and (4.26).

Since the coefficients a_n have no geometric meaning for even n, it makes sense to consider only the odd part of the Fourier series. Considering the odd part is the same as considering the Fourier series twisted by a Dirichlet character of modulus 2 (see **Section 4.4**). It is then natural to expect that, similarly to the case of modular forms (see [7, Theorem 7.4]), the odd part of the Fourier series is a quasimodular form on the congruence subgroup $\Gamma_0(4)$. Actually, we will prove this is the case. Let Φ_2 and Φ_4 be the two modular forms of respective weights 2 and 4, defined on $\Gamma_0(4)$ as in (4.27).

Theorem 1.5. The Fourier series

$$\sum_{n\in 2\mathbb{Z}_{\geq 0}+1} a_n \exp(2i\pi nz) \tag{1.4}$$

is the quasimodular form of weight 4 and depth 1 on $\Gamma_0(4)$ defined by

$$\frac{1}{1280}\left[E_4(z) - 9E_4(2z) + 8E_4(4z) - 15\frac{d}{2i\pi\, dz}\Phi_2(z) + 15\frac{d}{2i\pi\, dz}\Phi_4(z) \right]. \tag{1.5}$$

$\qquad\square$

Remark 1.6. This theorem will be proved in **Section 6**. It is interesting to note that forgetting the artificial terms of even order results in a lesser depth, that is, in a simplified modular situation. (A modular form is a quasimodular form of depth 0, so the depth may be seen as a measure of complexity.)

Table 1.1 gives the first few values of a_n^p and a_n.

Our results may be interpreted in terms of counting genus 2 covers of the torus $\mathsf{T} = \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ with one double ramification point (see **Section 2**). The general problem of counting covers with fixed ramification type of a given Riemann surface was posed in 1891 by Hurwitz who precisely counted the covers of the sphere. Dijkgraaf [2] computed

Table 1.1   Number of surfaces of type A.

n	5	7	9	11	13	15	17	19	21	23	25	27
a_n^p	18	54	108	225	378	504	864	1215	1440	2178	2700	3159
a_n	18	54	120	225	378	594	864	1215	1680	2178	2808	3630

the generating series of the countings of degree n and genus g covers of T with simple ramification over distinct points, weighted by the inverse of the number of automorphisms. Kaneko and Zagier [9] introduced the notion of quasimodular forms and proved that the generating series computed by Dijkgraaf was quasimodular of weight $6g - 6$ on $\mathrm{SL}(2, \mathbb{Z})$. The case of arbitrary ramification over a single point was studied by Bloch and Okounkov [1]. They proved that the countings lead to linear combinations of quasimodular forms of weight less than or equal to $6g - 6$. This was used by Eskin and Okounkov [4] to compute volumes of the strata of moduli spaces of translation surfaces (see also [15]). The $\mathrm{SL}(2, \mathbb{Z})$ orbits of square-tiled surfaces were studied by Hubert and Lelièvre in the case of a prime number of squares [6] and by McMullen [12] in the general case.

Up to a multiplicative constant factor, our counting functions are the orbifold Euler characteristics of Teichmüller curves. Matt Bainbridge independently obtained results similar to ours in this setting.

The moduli space of holomorphic 1-forms on complex curves of a fixed genus g can be considered as a family of flat structures of a special type on a surface of genus g. The group $\mathrm{GL}(2, \mathbb{R})$ acts naturally on the moduli space; its orbits, called Teichmüller discs, project to the moduli space of curves as complex geodesics for the Teichmüller metric. A typical flat surface has no symmetry; its stabilizer in $\mathrm{GL}(2, \mathbb{R})$ is trivial; the corresponding Teichmüller disc is dense in the moduli space. For some flat surfaces (called Veech surfaces) the stabiliser is big (a lattice) so that the corresponding Teichmüller disc is closed. Projections of such Teichmüller discs, called Teichmüller curves, play the role of "closed complex geodesics."

The main lines of the proof of Theorem 1.1 are the following. In Section 3, we evaluate the number a_n^p in terms of sums over sets defined by complicated arithmetic conditions. In Section 5, we relate these coefficients a_n^p to sums of sums of divisors of the form

$$\sum_{\substack{(a,b)\in\mathbb{Z}_{>0}^2 \\ ka+b=n}} \sigma_1(a)\sigma_1(b). \tag{1.6}$$

For the computation of these sums, we use, in Section 4, the notion of quasimodular forms on congruence subgroups (introduced by Kaneko and Zagier in [9]) and we take
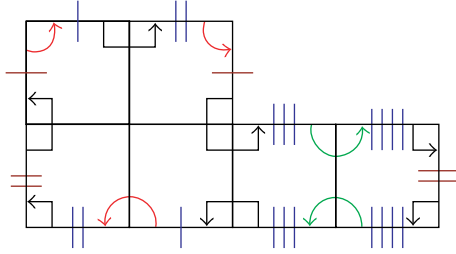
Figure 2.1   Surface with one angle of 6π.

advantage of the fact that the spaces of quasimodular forms have finite dimension to linearise the above sums. Here, linearising means expressing them as linear combinations of sums of powers of divisors. Having obtained a series whose odd coefficients are the numbers a_n, we introduce the notion of twist of a quasimodular form by a Dirichlet character, to construct a new quasimodular form generating series without artificial Fourier coefficients.

2   Geometric background

2.1   Square-tiled surfaces

A square-tiled surface is a collection of unit squares endowed with identifications of opposite sides: each top side is identified to a bottom side and each right side is identified to a left side. In addition, the resulting surface is required to be connected. A square-tiled surface tiled by n squares is also a degree n (connected) branched cover of the standard torus $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ with a single branch point.

Given a square-tiled surface, to each vertex can be associated an angle which is a multiple of 2π (four or a multiple of four squares can abutt at each vertex). If $(k_i + 1)2\pi$ is the angle at vertex i, the Gauss-Bonnet formula implies that

$$\sum_{i=1}^{s} k_i = 2g - 2, \tag{2.1}$$

where g is the genus of the surface and s the total number of vertices.

Figure 2.1 shows a surface where one vertex has angle 6π (and two other vertices have angle 2π).

Surfaces can be sorted according to strata $\mathcal{H}(k_1, \ldots, k_s)$. Square-tiled surfaces are integer points of these strata. In this paper, we are concerned with surfaces in $\mathcal{H}(2)$,
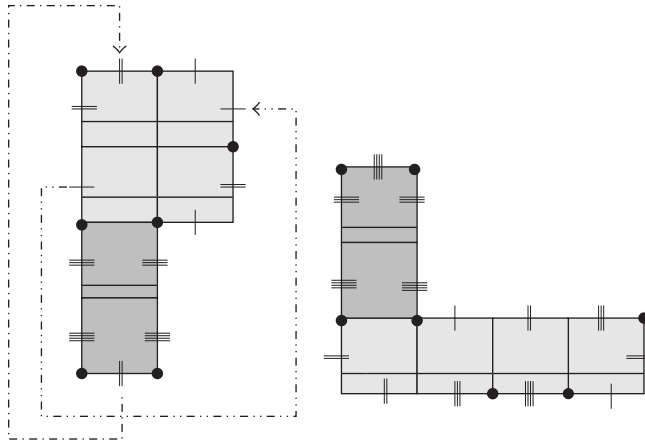
Figure 2.2   Decomposition of a surface into two cylinders.

that is, with a single ramification point of angle 6π. A surface tiled by n squares in $\mathcal{H}(2)$ is a degree n branched cover of the torus $\mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ with one double ramification point.

2.2   Cylinder decompositions

Given any square-tiled surface, each horizontal line on the surface through the interior of a square is closed, and neighbouring horizontal lines are also closed. Thus closed horizontal lines come in families forming cylinders and the surface decomposes into such cylinders bounded by horizontal saddle connections (segments joining conical singularities). This is illustrated in Figure 2.2.

Here we explain how to enumerate square-tiled surfaces in $\mathcal{H}(2)$ with a given number of squares, by giving a system of coordinates for them. We include this discussion for the sake of completeness, although these coordinates have already been used in [3, 6, 15].

We represent surfaces according to their cylinder decompositions. Cylinders of a square-tiled surface are naturally represented as rectangles. One can cut a triangle from one side of such a rectangle and glue it back on the other side according to the identifications to produce a parallelogram with a pair of horizontal sides (each made of one or several saddle connections), and a pair of identified nonhorizontal parallel sides. A square-tiled surface in $\mathcal{H}(2)$ has one or two cylinders [15] and can always be represented as in Figure 2.3 or 2.4. Each cylinder has a height and a width and in addition a twist parameter corresponding to the possibility of rotating the saddle connections of the top or bottom of the cylinders before performing the identifications.
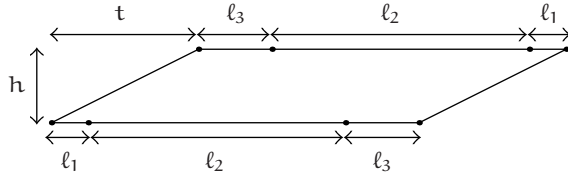
Figure 2.3   One-cylinder surface.

2.3   One-cylinder surfaces

For one-cylinder surfaces in $\mathcal{H}(2)$, we have on the bottom of the cylinder three horizontal saddle connections, and the same saddle connections appear on the top of the cylinder in reverse order; we denote by ℓ the width of the cylinder and ℓ_1, ℓ_2, ℓ_3 the lengths of the saddle connections, numbered so that they appear in that order on the bottom side and in reverse order on the top side. See Figure 2.3.

For each choice of (ℓ_1, ℓ_2, ℓ_3) with $\ell_1 + \ell_2 + \ell_3 = \ell$, if ℓ_1, ℓ_2, ℓ_3 are not all equal, there are ℓ possible values of the twist t giving different surfaces. But, the three possible cyclic permutations of (ℓ_1, ℓ_2, ℓ_3) yield the same set of surfaces. So, to make coordinates uniquely defined, we require that (ℓ_1, ℓ_2, ℓ_3) has least lexicographic order among its cyclic permutations. For countings, it is simpler to ignore this point and to divide by 3 at the end.

If ℓ_1, ℓ_2, ℓ_3 are all equal (and thus worth $\ell/3$), there is only one cyclic permutation of (ℓ_1, ℓ_2, ℓ_3) but only $\ell/3$ values of the twist t give different surfaces.

The parameters we have used satisfy

$$\begin{aligned} &\ell \mid n, \\ &\ell_1 + \ell_2 + \ell_3 = \ell, \\ &0 \leq t < \ell \qquad \text{or} \qquad \frac{\ell}{3}. \end{aligned} \tag{2.2}$$

Remark 2.1. From this description of coordinates, we conclude that the number of one-cylinder surfaces in $\mathcal{H}(2)$ tiled with n squares is (see [3])

$$\frac{1}{3} \sum_{\ell \mid n} \sum_{\substack{(\ell_1, \ell_2, \ell_3) \in \mathbb{Z}_{>0}^3 \\ \ell_1 + \ell_2 + \ell_3 = \ell}} \ell. \tag{2.3}$$

2.4   Two-cylinder surfaces

Given a two-cylinder surface in $\mathcal{H}(2)$, one of its cylinders (call it cylinder 1) has one saddle connection on the top and one saddle connection (of same length) on the bottom,
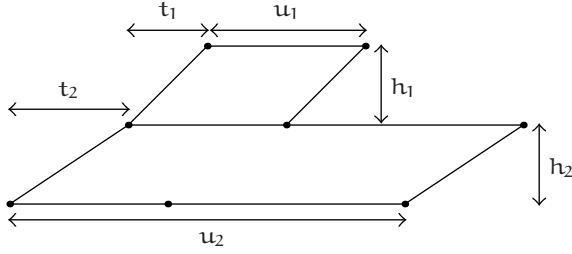
Figure 2.4   Two-cylinder surface.

while the other one (call it cylinder 2) is bounded by two saddle connections on the top and two saddle connections on the bottom. See Figure 2.4.

For each of cylinders 1 and 2, there are three parameters: the height h_i, the width u_i, and the twist t_i.

Given two heights h_1 and h_2, two widths $u_1 < u_2$, and two twists t_1, t_2 with $0 \leq t_i < u_i$, there exists a unique surface in $\mathcal{H}(2)$ with two cylinders having $(h_1, h_2, u_1, u_2, t_1, t_2)$ as parameters. The number of squares is then $h_1 u_1 + h_2 u_2$.

Remark 2.2. From this system of coordinates one deduces (see [3]) that the number of two-cylinder surfaces in $\mathcal{H}(2)$ tiled by n squares is

$$\sum_{\substack{(h_1, h_2, u_1, u_2) \in \mathbb{Z}_{>0}^4 \\ u_1 < u_2 \\ h_1 u_1 + h_2 u_2 = n}} u_1 u_2. \tag{2.4}$$

2.5   Lattice of periods

The lattice of periods of a square-tiled surface is the rank two sublattice of $\mathbb{Z}^2$ generated by its saddle connections.

Lemma 2.3. A square-tiled surface is translation-tiled by a parallelogram if and only if this parallelogram is a fundamental domain for a lattice containing the surface's lattice of periods. $\qquad\square$

Proof. Decompose the surface into polygons with vertices at the conical singularities. The sides of these polygons are saddle connections and together generate the lattice of periods. The tiling of the plane by parallelograms which are a fundamental domain for this lattice (or any rank two lattice of the plane containing it) yields a tiling of the translation surface by such parallelograms. $\qquad\blacksquare$

Remark 2.4. The previous lemma implies that the area of the lattice of periods divides the area of the surface it comes from.

We will use a basis of the lattice of periods given by the following lemma [14, Chapter 7].

Lemma 2.5. Let Λ be a sublattice of $\mathbb{Z}+i\mathbb{Z}$ of index d. Then there exists a unique triple of integers (a, t, h) with $a \geq 1$, $ah = d$, and $0 \leq t \leq a - 1$ such that

$$\Lambda = (a, 0)\mathbb{Z} \oplus (t, h)\mathbb{Z}. \tag{2.5}$$

$\square$

Remark 2.6. Let S be a square-tiled surface in $\mathcal{H}(2)$, and let $\left(\begin{smallmatrix} a & t \\ 0 & h \end{smallmatrix}\right)$ be the matrix corresponding to its lattice of periods. If S is one-cylinder, then h is the height of its unique cylinder; if S is two-cylinder, with cylinders of height h_1 and h_2, then $h = (h_1, h_2)$.

Definition 2.7. A square-tiled surface is called *primitive* if its lattice of periods is $\mathbb{Z}^2$, in other words if $\left(\begin{smallmatrix} a & t \\ 0 & h \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Definition 2.8. A square-tiled surface is called *primitive in height* if $h = 1$.

The linear action of $\mathrm{GL}(2, \mathbb{Q})^+$ on $\mathbb{R}^2$ induces an action of $\mathrm{GL}(2, \mathbb{Q})^+$ on square-tiled surfaces. This action preserves orientation. The action of $\mathrm{SL}(2, \mathbb{Z})$ preserves the number of square tiles, and preserves primitivity.

Hubert and Lelièvre have shown that if $n \geq 5$ is prime, the square-tiled surfaces in $\mathcal{H}(2)$ tiled with n squares (necessarily primitive since n is prime) form two orbits under $\mathrm{SL}(2, \mathbb{Z})$, denoted by A_n and B_n.

If n is not prime and $n \geq 6$, not all surfaces tiled by n squares are primitive, and if n has many divisors, these surfaces split into many orbits under $\mathrm{SL}(2, \mathbb{Z})$, most of them lying in orbits under $\mathrm{GL}(2, \mathbb{Q})^+$ of primitive square-tiled surfaces with fewer squares. There can be an arbitrary number of such "artificial" $\mathrm{SL}(2, \mathrm{Z})$-orbits. Artificial orbits consist only of nonprimitive square-tiled surfaces, since the action of $\mathrm{SL}(2, \mathbb{Z})$ preserves primitivity.

Let n be an odd integer. We can distinguish two types of surfaces among surfaces tiled by n squares in $\mathcal{H}(2)$. These two types are distinguished by Weierstrass points, as follows (see [6]).

On a surface in $\mathcal{H}(2)$, the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ induces an involution which can be shown to have six fix points, called the Weierstrass points of the surface. It is easy to show that for a square-tiled surface these points have coordinates in $(1/2)\mathbb{Z}$. The type invariant is determined by the number of Weierstrass points which have both their coordinates

integer:

    (i) a surface is of type A if it has one integer Weierstrass point;

    (ii) a surface is of type B if it has three integer Weierstrass points.

Remark 2.9. We give an interpretation in terms of orbits. Consider the orbit under $GL(2, \mathbb{Q})^+$ of a surface S tiled by n squares. Then

    (i) the primitive square-tiled surfaces in this orbit all have the same number of squares, say d;

    (ii) the action of $GL(2, \mathbb{Q})^+$ restricts to an action of $SL(2, \mathbb{Z})$ on these primitive square-tiled surfaces;

    (iii) these primitive surfaces form an orbit under $SL(2, \mathbb{Z})$;

    (iv) McMullen extended the result of Hubert and Lelièvre by showing that if $d \geq 5$ is odd, the set of primitive square-tiled surfaces in $\mathcal{H}(2)$ tiled with d squares is partitioned in two orbits under $SL(2, \mathbb{Z})$ denoted by A_d and B_d;

    (v) the type can be read on these primitive square-tiled surfaces.

3 Sum-type formulae for the orbitwise countings

3.1 From primitive to nonprimitive countings

In this section, we establish relations between countings of primitive surfaces, countings of height-primitive surfaces, and countings of (non-necessarily primitive) surfaces. For any integer ℓ, the function σ_ℓ is defined by

$$\sigma_\ell(n) = \begin{cases} \displaystyle\sum_{d|n} d^\ell & \text{if } n \in \mathbb{Z}_{>0}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

For $n \in \mathbb{Z}_{>0}$, define E_n as the set of surfaces in $\mathcal{H}(2)$ tiled by n squares, E_n^p as its subset of primitive surfaces, and E_n^{ph} as its subset of primitive in height surfaces. For $d \in \mathbb{Z}_{>0}$, denote Λ_d the set of sublattices of $\mathbb{Z} + i\mathbb{Z}$ of index d. The description of surfaces by primitive surfaces is given by the following lemma.

Lemma 3.1. For $n \in \mathbb{Z}_{>0}$, the following bijection holds:

$$E_n \simeq \bigcup_{d|n} E_{n/d}^p \times \Lambda_d. \tag{3.2}$$
$$\square$$

Proof. Let $S \in E_n$ and let d be the index in $\mathbb{Z} + i\mathbb{Z}$ of its lattice of periods $\mathrm{Per}(S)$. Then $d \mid n$ and $\mathrm{Per}(S) \in \Lambda_d$. With the notations of **Lemma 2.5**, we write $\mathrm{Per}(S) = (a, 0)\mathbb{Z} \oplus (t, h)\mathbb{Z}$. To

S we associate a surface tiled by n/d squares:

$$S' = \begin{pmatrix} a & t \\ 0 & h \end{pmatrix}^{-1} S. \tag{3.3}$$

The lattice of periods of S' is $\mathbb{Z} + i\mathbb{Z}$ so that it is primitive. Conversely, let $S' \in E^p_{n/d}$ and $\Lambda \in \Lambda_d$. With the notations of Lemma 2.5, we write $\Lambda = (a, 0)\mathbb{Z} \oplus (t, h)\mathbb{Z}$. Then

$$S = \begin{pmatrix} a & t \\ 0 & h \end{pmatrix} S' \tag{3.4}$$

has $n = ah$ squares. ∎

Corollary 3.2. For $n \in \mathbb{Z}_{>0}$,

$$\#E_n = \sum_{d|n} \sigma_1(d) \#E^p_{n/d}. \tag{3.5}$$
□

Proof. By Lemma 2.5, we have

$$\#\Lambda_d = \sum_{\substack{(a,t,h) \in \mathbb{Z}^3_{\geq 0} \\ ah = d \\ 0 \leq t < a}} 1 = \sigma_1(d). \tag{3.6}$$
∎

We recall that a surface S is primitive in height if $h = 1$ with the notations of Lemma 2.5. That is, its lattice of periods is $\text{Per}(S) = (a, 0)\mathbb{Z} + (t, 1)\mathbb{Z}$ with $a \geq 1$ and $0 \leq t \leq a - 1$. We write Λ'_d for the set of these lattices having index d (implying $d = a$). We have $\#\Lambda'_d = d$. Similarly to Lemma 3.1, we have the following.

Lemma 3.3. For $n \in \mathbb{Z}_{>0}$, the following bijection holds:

$$E^{ph}_n \simeq \bigcup_{d|n} E^p_{n/d} \times \Lambda'_d. \tag{3.7}$$
□

Corollary 3.4. For $n \in \mathbb{Z}_{>0}$,

$$\#E^{ph}_n = \sum_{d|n} d \cdot \#E^p_{n/d}. \tag{3.8}$$
□

We deduce the same result for surfaces of type A. For odd n, define A'_n as the set of n-square-tiled surfaces of type A in $\mathcal{H}(2)$, A^p_n as its subset of primitive surfaces (which coincides with the $SL(2, \mathbb{Z})$-orbit A_n), and A^{ph}_n as its subset of height-primitive surfaces.

Lemma 3.5. For $n \in \mathbb{Z}$ odd, the following bijection holds:

$$A'_n \simeq \bigcup_{d|n} A^p_{n/d} \times \Lambda_d. \tag{3.9}$$

$\square$

Proof. We recall that the type of a surface is characterized by the number of its Weierstrass points with integer coordinates. To deduce **Lemma 3.5** from **Lemma 3.1** it then suffices to prove that a Weierstrass point P has half-integer coordinates[1] in a basis determined by $\mathrm{Per}(S)$ if and only if its image by the bijection of **Lemma 3.1** has half-integer coordinates in the canonical basis of $\mathbb{Z} + i\mathbb{Z}$. Let $S \in E_n$, $\mathrm{Per}(S) = (a, 0)\mathbb{Z} \oplus (t, h)\mathbb{Z}$ its lattice of periods with the notations of **Lemma 2.5**. We set

$$M = \begin{pmatrix} a & t \\ 0 & h \end{pmatrix}. \tag{3.10}$$

Let P a Weierstrass point in S, we assume that its coordinates in the basis of $\mathrm{Per}(S)$ are $(\ell/2, m/2)$ with m and n not simultaneously even. The coordinates of P in $\mathbb{Z} + i\mathbb{Z}$ are therefore $(a\ell + mt, mh)/2$, hence those of $M^{-1}P$ in $M^{-1}S$ are $(\ell/2, m/2)$ in the standard basis of $\mathbb{Z} + i\mathbb{Z}$. ∎

Corollary 3.6. For $n \in \mathbb{Z}_{>0}$,

$$a_n = \sum_{d|n} \sigma_1(d) a^p_{n/d}. \tag{3.11}$$

$\square$

Lemma 3.7. For $n \in \mathbb{Z}$ odd, the following bijection holds:

$$A^{ph}_n \simeq \bigcup_{d|n} A^p_{n/d} \times \Lambda'_d. \tag{3.12}$$

$\square$

Corollary 3.8. For $n \in \mathbb{Z}_{>0}$,

$$a^{ph}_n = \sum_{d|n} d a^p_{n/d}. \tag{3.13}$$

$\square$

To express the number of primitive surfaces in terms of the numbers of primitive in height ones, we recall some basic facts on L-functions. For an arithmetic function f,

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[1] Meaning in $(1/2)\mathbb{Z}^2$ but not in $\mathbb{Z}^2$.

we define

$$L(f, s) = \sum_{n=1}^{+\infty} f(n)n^{-s}. \tag{3.14}$$

If id denotes the identity function, we have

$$L\left(\mathrm{id}^{\ell} f, s\right) = L(f, s - \ell). \tag{3.15}$$

For f and g two arithmetic functions with convolution product $f * g$, we have

$$L(f * g, s) = L(f, s)L(g, s). \tag{3.16}$$

The constant equal to 1 function is denoted by   and we have

$$L(\ , s) = \zeta(s) \tag{3.17}$$

the Riemann ζ function. Moreover

$$L(\mu, s) = \frac{1}{\zeta(s)}, \qquad L\left(\sigma_k, s\right) = \zeta(s)\zeta(s - k). \tag{3.18}$$

Lemma 3.9. Let $n \in \mathbb{Z}_{>0}$. Then

$$a_n^p = \sum_{d|n} d\mu(d)a_{n/d}^{ph}. \tag{3.19}$$
$$\square$$

Proof. **Lemma 3.7** is then

$$L\left(a^{ph}, s\right) = \zeta(s - 1)L\left(a^p, s\right). \tag{3.20}$$

We deduce

$$L\left(a^p, s\right) = L(\mu, s - 1)L\left(a^{ph}, s\right) = L(\mu\,\mathrm{id}, s)L\left(a^{ph}, s\right), \tag{3.21}$$

hence the result.   ■

Next, we give sum-type formulae for the number of surfaces in A_n^{ph}.

Proposition 3.10. Let $n \in \mathbb{Z}_{>0}$, the number of height-primitive one-cylinder surfaces with n squares in $\mathcal{H}(2)$ of type A is

$$\frac{1}{3} \sum_{\substack{\ell_1, \ell_2, \ell_3 \text{ odd} \\ \ell_1 + \ell_2 + \ell_3 = n}} n. \tag{3.22}$$
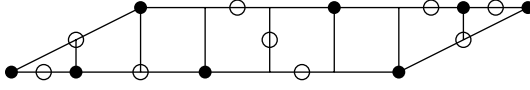$$\square$$

Figure 3.1   Weierstrass points of a one-cylinder surface.

Proof. See Figure 3.1. Since the cylinder is primitive in height, it has height 1. As proved in [6, Section 5.1.1], the Weierstrass points are

    (i) the saddle point, which has integer coordinates,

   (ii) two points lying on the core of the cylinder, which do not have integer coordinates,

  (iii) the midpoints of the three saddle connections, each of these points having integer coordinates if and only if the corresponding saddle connection has even length.   ■

Proposition 3.11. Let $n \in \mathbb{Z}_{>0}$, the number of height-primitive two-cylinder surfaces with n squares in $\mathcal{H}(2)$ of type A is

$$\sum_{\substack{h_1,h_2,u_1,u_2 \in \mathbb{Z}_{\geq 0} \\ h_1 u_1 + h_2 u_2 = n \\ (h_1,h_2)=1 \\ h_1,h_2 \text{ odd} \\ u_1 < u_2}} u_1 u_2 + \frac{1}{2} \sum_{\substack{h_1,h_2,u_1,u_2 \in \mathbb{Z}_{\geq 0} \\ h_1 u_1 + h_2 u_2 = n \\ (h_1,h_2)=1 \\ h_1 \not\equiv h_2 \,(\text{mod } 2) \\ u_1 < u_2 \\ u_1 u_2 \text{ even}}} u_1 u_2. \tag{3.23}$$

       □

Proof. See Figure 3.2. Among height-primitive two-cylinder surfaces with parameters $h_1, h_2, u_1, u_2, t_1, t_2$, such that $h_1 u_1 + h_2 u_2 = n$ (odd):

    (i) all surfaces with h_1 and h_2 odd are of type A;

   (ii) all surfaces with u_1 and u_2 odd are of type B;

  (iii) exactly half of the remaining surfaces (with different parity for u_1 and u_2 and for h_1 and h_2) are of type A, and half are of type B;

for each (h_1, h_2, u_1, u_2), there are $u_1 u_2$ possible twists (if n is not prime, some values of the twists may yield nonprimitive surfaces which is why we only require height-primitivity). In the case of different parities for h_1 and h_2 and for u_1 and u_2, the product $u_1 u_2$ is even and exactly half of the possible twists correspond to each type. The height-primitivity condition is just that the heights of the cylinders have greatest common divisor equal to one.   ■
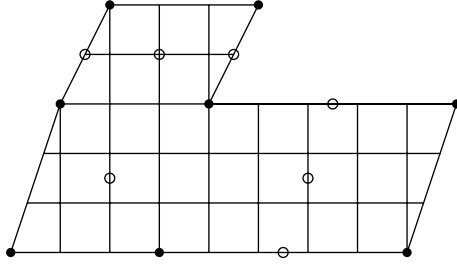
Figure 3.2   Weierstrass points of a two-cylinder surface.

4   Quasimodular forms

4.1   Motivation

The aim of this part is the computation of sums of type

$$S_k(n) = \sum_{\substack{(a,b)\in\mathbb{Z}_{>0}^2 \\ ka+b=n}} \sigma_1(a)\sigma_1(b) \tag{4.1}$$

with $k \in \mathbb{Z}_{>0}$. Here we study only the cases $k \in \{1,2,4\}$ but the method in fact applies to every k [13].

Useful to the study of these sums is the weight 2 Eisenstein series

$$E_2(z) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n)e(nz), \tag{4.2}$$

where

$$e(\tau) = \exp(2i\pi\tau) \quad (\Im m\,\tau > 0). \tag{4.3}$$

Defining

$$H_k(z) = E_2(z)E_2(kz), \tag{4.4}$$

one gets

$$H_k(z) = 1 - 24 \sum_{n=1}^{+\infty} \left[\sigma_1(n) + \sigma_1\left(\frac{n}{k}\right)\right]e(nz) + 576 \sum_{n=1}^{+\infty} S_k(n)e(nz). \tag{4.5}$$

We will achieve the linearisation of H_k using the theory of quasimodular forms, developed by Kaneko and Zagier. The computation of $S_k(n)$ will be deduced for each n.

4.2   Definition

Let us therefore begin by surveying our prerequisites on quasimodular forms, referring to [10, Section 17] for the details. Define

$$
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (a, b, c, d) \in \mathbb{Z}^4, \ ad - bc = 1, \ N \mid c \right\} \tag{4.6}
$$

for all integers $N \geq 1$. In particular, $\Gamma_0(1)$ is $SL(2, \mathbb{Z})$. Denote by $\mathcal{H}$ the Poincaré upper half plane:

$$
\mathcal{H} = \{z \in \mathbb{C} : \Im m \, z > 0\}. \tag{4.7}
$$

Definition 4.1. Let $N \in \mathbb{Z}_{>0}$, $k \in \mathbb{Z}_{\geq 0}$, and $s \in \mathbb{Z}_{\geq 0}$. A holomorphic function

$$
f : \mathcal{H} \longrightarrow \mathbb{C} \tag{4.8}
$$

is a quasimodular form of weight k and depth s on $\Gamma_0(N)$ if there exist holomorphic functions $f_0, f_1, \dots, f_s$ on $\mathcal{H}$ such that

$$
(cz + d)^{-k} f\left(\frac{az + b}{cz + d} \right) = \sum_{i=0}^{s} f_i(z) \left(\frac{c}{cz + d} \right)^i \tag{4.9}
$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and such that f_s is holomorphic at the cusps and not identically vanishing. By convention, the 0 function is a quasimodular form of depth 0 for each weight.

Here is what is meant by the requirement for f_s to be holomorphic at the cusps. One can show [10, Lemma 119] that if f satisfies the quasimodularity condition (4.9), then f_s satisfies the modularity condition

$$
(cz + d)^{-(k-2s)} f_s\left(\frac{az + b}{cz + d} \right) = f_s(z) \tag{4.10}
$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Asking f_s to be holomorphic at the cusps is asking that, for all $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(1)$, the function

$$
z \longmapsto (\gamma z + \delta)^{-(k-2s)} f_s\left(\frac{\alpha z + \beta}{\gamma z + \delta} \right) \tag{4.11}
$$

has a Fourier expansion of the form

$$\sum_{n=0}^{+\infty} \widehat{f}_{s,M}(n) e\left(\frac{nz}{u_M}\right), \tag{4.12}$$

where

$$u_M = \inf\left\{u \in \mathbb{Z}_{>0} : T^u \in M^{-1}\Gamma_0(N)M\right\}. \tag{4.13}$$

In other words, f_s is automatically a modular function and is required to be more than that, a modular form of weight $k - 2s$ on $\Gamma_0(N)$. It follows that if f is a quasimodular form of weight k and depth s, non-identically vanishing, then k is even and $s \le k/2$.

Remark 4.2. Let χ be a Dirichlet character (see Section 4.4). If f satisfies all of what is needed to be a quasimodular form except (4.9) being replaced by

$$(cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right) = \chi(d) \sum_{i=0}^{n} f_i(z)\left(\frac{c}{cz + d}\right)^i, \tag{4.14}$$

then one says that f is a quasimodular form of weight k, depth s, and character χ on $\Gamma_0(N)$.

The Eisenstein series E_2 transforms as

$$(cz + d)^{-2} E_2\left(\frac{az + b}{cz + d}\right) = E_2(z) + \frac{6}{i\pi} \frac{c}{cz + d} \tag{4.15}$$

under the action of any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(1)$. Hence, E_2 is a quasimodular form of weight 2 and depth 1 on $\Gamma_0(1)$. Defining

$$E_{N,2}(z) = E_2(Nz), \tag{4.16}$$

one has

$$(cz + d)^{-2} E_{N,2}\left(\frac{az + b}{cz + d}\right) = E_{N,2}(z) + \frac{6}{i\pi N} \frac{c}{cz + d} \tag{4.17}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Hence, $E_{N,2}$ is a quasimodular form of weight 2 and depth 1 on $\Gamma_0(N)$. One denotes by $\widetilde{M}_k[\Gamma_0(N)]^{\le s}$ the space of quasimodular forms of weight k and depth less than or equal to s on $\Gamma_0(N)$. The space $\widetilde{M}_k[\Gamma_0(N)]^{\le 0}$ is the space $M_k[\Gamma_0(N)]$ of modular forms of weight k on $\Gamma_0(N)$. A recurrence on the depth implies [10, Corollary 121]

the equality

$$\widetilde{M}_k\big[\Gamma_0(N)\big]^{\leq s} = \bigoplus_{i=0}^{s} M_{k-2i}\big[\Gamma_0(N)\big] E_2^i. \tag{4.18}$$

It is known that $M_2[\Gamma_0(1)] = \{0\}$. However, if $N > 1$, one deduces from

$$\mathbb{C}E_2 \oplus \mathbb{C}E_{N,2} \subset \widetilde{M}_2\big[\Gamma_0(N)\big]^{\leq 1} = M_2\big[\Gamma_0(N)\big] \oplus \mathbb{C}E_2 \tag{4.19}$$

that $\dim M_2[\Gamma_0(N)] \geq 1$. By the way, for every family $(c_d)_{d|N}$ such that

$$\sum_{d|N} \frac{c_d}{d} = 0, \tag{4.20}$$

one has

$$\left[z \longmapsto \sum_{d|N} c_d E_2(dz)\right] \in M_2\big[\Gamma_0(N)\big]. \tag{4.21}$$

Denote by D the differential operator

$$D = \frac{1}{2i\pi}\frac{d}{dz}. \tag{4.22}$$

It defines a linear application from $\widetilde{M}_k[\Gamma_0(N)]^{\leq s}$ to $\widetilde{M}_{k+2}[\Gamma_0(N)]^{\leq s+1}$. This application is injective and strictly increases the depth if $k > 0$. This property allows to linearise the basis given in (4.18).

Lemma 4.3. Let $k \geq 2$ even. Then

$$\widetilde{M}_k\big[\Gamma_0(N)\big]^{\leq k/2} = \bigoplus_{i=0}^{k/2-1} D^i M_{k-2i}\big[\Gamma_0(N)\big] \oplus \mathbb{C}D^{k/2-1}E_2. \tag{4.23}$$
$\square$

4.3 Sums of sums of divisors

Lemma 4.3 allows to reach our goal by expressing the sums S_1, S_2, and S_4 introduced in (4.1) as follows.

Proposition 4.4. Let $n \geq 1$. Then

$$S_1(n) = \frac{5}{12}\sigma_3(n) - \frac{n}{2}\sigma_1(n) + \frac{1}{12}\sigma_1(n),$$

$$S_2(n) = \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{8}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{2}\right) + \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{2}\right),$$

$$S_4(n) = \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{4}\right) - \frac{1}{16}n\sigma_1(n) - \frac{1}{4}n\sigma_1\left(\frac{n}{4}\right)$$
$$+ \frac{1}{24}\sigma_1(n) + \frac{1}{24}\sigma_1\left(\frac{n}{4}\right).$$

$$(4.24)$$

$\square$

Proof. We detail the proof for the expression of S_4. The function H_4, introduced in (4.5), is a quasimodular form of weight 4 and depth 2 on $\Gamma_0(4)$. Lemma 4.3 gives

$$\widetilde{M}_4\left[\Gamma_0(4)\right]^{\leq 2} = M_4\left[\Gamma_0(4)\right] \oplus DM_2\left[\Gamma_0(4)\right] \oplus \mathbb{C}DE_2. \tag{4.25}$$

The space $M_4[\Gamma_0(4)]$ has dimension 3 and contains the linearly independent functions

$$E_4(z) = 1 + 240\sum_{n=1}^{+\infty}\sigma_3(n)e(nz),$$
$$E_{2,4}(z) = E_4(2z),$$
$$E_{4,4}(z) = E_4(4z).$$

$$(4.26)$$

The space $M_2[\Gamma_0(4)]$ has dimension 2 and is generated by

$$\Phi_2(z) = 2E_2(2z) - E_2(z),$$
$$\Phi_4(z) = \frac{4}{3}E_2(4z) - \frac{1}{3}E_2(z).$$

$$(4.27)$$

Hence, by the computations of the first seven Fourier coefficients, one gets

$$H_4 = \frac{1}{20}E_4 + \frac{3}{20}E_{2,4} + \frac{4}{5}E_{4,4} + \frac{9}{2}D\Phi_4 + 3DE_2. \tag{4.28}$$

The computation of S_4 is then obtained by comparison of the Fourier coefficients of this equality. The computation of S_2 is obtained along the same lines via the equality

$$H_2 = \frac{1}{5}E_4 + \frac{4}{5}E_{2,4} + 3D\Phi_2 + 6DE_2 \tag{4.29}$$

between forms of $\widetilde{M}_4[\Gamma_0(2)]^{\leq 2}$. At last, the expression of S_1 is deduced from the equality

$$E_2^2 = E_4 + 12DE_2 \tag{4.30}$$

between forms of $\widetilde{M}_4[\Gamma_0(1)]^{\leq 2}$. ∎

Remark 4.5. The computation of H_4, which lies in the dimension 6 vector space with basis $\{E_4, E_{2,4}, E_{4,4}, D\Phi_2, D\Phi_4, DE_2\}$, required working on *seven* consecutive Fourier coefficients. We briefly explain why, mentioning that any sequence of 6 consecutive coefficients is not sufficient. For any function

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n)e(nz) \tag{4.31}$$

and any integer $i \geq 0$, define

$$c(f, i) = \left(\widehat{f}(i), \widehat{f}(i+1), \widehat{f}(i+2), \widehat{f}(i+3), \widehat{f}(i+4), \widehat{f}(i+5) \right), \tag{4.32}$$

and let

$$\begin{aligned}
v_1(i) &= c(E_4, i), & v_2(i) &= c(E_{2,4}, i), & v_3(i) &= c(E_{4,4}, i), \\
v_4(i) &= c(D\Phi_2, i), & v_5(i) &= c(D\Phi_4, i), & v_6(i) &= c(DE_2, i).
\end{aligned} \tag{4.33}$$

Then, for each i, there exists an explicitly computable linear relation between $v_2(i)$, $v_3(i)$, $v_4(i)$, $v_5(i)$, and $v_6(i)$. One could think of using a basis of $\widetilde{M}_4[\Gamma_0(1)]^{\leq 2}$ echelonized by increasing powers of $e(z)$. The same phenomenon would however appear when changing to such a basis and expressing the new basis elements in terms of the original basis.

4.4   Twist by a Dirichlet character

Recall that a Dirichlet character χ is a character of a multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$ extended to a function on $\mathbb{Z}$ by defining

$$\chi(n) = \begin{cases} \chi(n \,(\text{mod } q)) & \text{if } (n, q) = 1, \\ 0 & \text{otherwise} \end{cases} \tag{4.34}$$

(see, e.g., [8, Chapter 3]).

A quasimodular form admits a Fourier expansion

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n)e(nz). \tag{4.35}$$

Since we will need to compute the odd part of a quasimodular form, we introduce the notion of twist of a quasimodular form by a Dirichlet character.

Definition 4.6. Let χ be a Dirichlet character. Let f be a function having Fourier expansion of the form (4.35). The twist of f by χ is the function $f \otimes \chi$ defined by the Fourier expansion

$$f \otimes \chi(z) = \sum_{n=0}^{+\infty} \chi(n) \widehat{f}(n) e(nz). \tag{4.36}$$

The interest of this definition is that it allows to build quasimodular forms, as stated in the next proposition.

Proposition 4.7. Let χ be a Dirichlet character of conductor m with nonvanishing Gauss sum. Let f be a quasimodular form of weight k and depth s on $\Gamma_0(N)$. Then $f \otimes \chi$ is a quasimodular form of weight k, depth less than or equal s, and character χ^2 on $\Gamma_0(\mathrm{lcm}(N, m^2))$.

$\square$

Remark 4.8. The Gauss sum of a character χ modulo m is defined by

$$\tau(\chi) = \sum_{u \,(\mathrm{mod}\, m)} \chi(u) e\left(\frac{u}{m}\right). \tag{4.37}$$

Proof. The proof is an adaptation of the corresponding result for modular forms (see, e.g., [7, Theorem 7.4]). We consider for each k the following action of $SL_2(\mathbb{R})$ on holomorphic functions on $\mathcal{H}$:

$$\left(f \mid_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = (cz + d)^{-k} f\left(\frac{az + b}{cz + d}\right). \tag{4.38}$$

Since χ is primitive, this sum is not zero. One has

$$\tau(\overline{\chi}) g \otimes \chi = \sum_{v \,(\mathrm{mod}\, m)} \overline{\chi}(v) \left(g \mid_k \begin{pmatrix} 1 & \frac{v}{m} \\ 0 & 1 \end{pmatrix}\right) \tag{4.39}$$

as soon as g has a Fourier expansion of the form (4.35). Define $M = \mathrm{lcm}(N, m^2)$. Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(M)$. The matrix

$$\begin{pmatrix} 1 & \frac{v}{m} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & \frac{v\delta^2}{m} \\ 0 & 1 \end{pmatrix}^{-1} \tag{4.40}$$

being in $\Gamma_0(N)$, one deduces from the level N quasimodularity of f and (4.39) that

$$\tau(\overline{\chi})\left(f\otimes\chi\,\Big|_k\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\right)(z)$$

$$=\sum_{i=0}^s\sum_{v\,(\mathrm{mod}\,m)}\overline{\chi}(v)\left(f_i\,\Big|_{k-2i}\begin{pmatrix}1&\frac{\delta^2 v}{m}\\0&1\end{pmatrix}\right)(z)\left[\frac{\gamma}{\gamma\left(z+\frac{\delta^2 v}{m}\right)+\delta-\frac{\gamma\delta^2 v}{m}}\right]^i.$$

$$(4.41)$$

Since the functions f_i are themselves quasimodular forms (see [10, Lemma 119]), they admit a Fourier expansion. Hence, from (4.39),

$$\tau(\overline{\chi})\left(f\otimes\chi\,\Big|_k\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\right)(z)=\tau(\overline{\chi})\chi(\delta)^2\sum_{i=0}^s f_i\otimes\chi(z)\left(\frac{\gamma}{\gamma z+\delta}\right)^i.\qquad(4.42)$$

It follows that $f\otimes\chi$ satisfies the quasimodularity condition. There remains to prove the holomorphy at the cusps, which is quite delicate since $f_s\otimes\chi$ may be 0 even though f_s is not. Actually, Lemma 4.3 and the fact that the twist of a modular form on $\Gamma_0(N)$ by a primitive Dirichlet character of conductor m is a modular form on $\Gamma_0(M)$ show that the proposition is proved as soon as it is proved for $f=D^{k/2-1}E_2$. In that case, $s=k/2$ and $f_s\otimes\chi$ is not 0 (see [10, Lemma 118]), hence f_s being a modular form implies that $f_s\otimes\chi$ is also one. ∎

5   Proof of Hubert and Lelièvre conjecture

The aim of this part is the proof of Theorem 1.1. In all this part, n is assumed to be odd. Define

$$\alpha_1(n,r)=\sum_{(h_1,u_1,h_2,u_2)\in\mathcal{A}_1(n,r)}u_1 u_2,$$

$$\alpha_2(n,r)=\frac{1}{2}\sum_{(h_1,u_1,h_2,u_2)\in\mathcal{A}_2(n,r)}u_1 u_2,\qquad(5.1)$$

$$\alpha_3(n,r)=\frac{n}{3}\sum_{(u_1,u_2,u_3)\in\mathcal{A}_3(n,r)}1$$

with

$\mathcal{A}_1(n, r)$

$$= \left\{ (h_1, u_1, h_2, u_2) \in \mathbb{Z}_{>0}^4 : \left| \begin{array}{l} (h_1, h_2) = 1, \\ h_1 \text{ and } h_2 \text{ odd,} \end{array} \right. u_1 < u_2, \; h_1 u_1 + h_2 u_2 = \frac{n}{r} \right\},$$

$\mathcal{A}_2(n, r)$

$$= \left\{ (h_1, u_1, h_2, u_2) \in \mathbb{Z}_{>0}^4 : \left| \begin{array}{l} (h_1, h_2) = 1, \\ h_1 \text{ or } h_2 \text{ even,} \end{array} \right| \begin{array}{l} u_1 < u_2, \\ u_1 \text{ or } u_2 \text{ even,} \end{array} h_1 u_1 + h_2 u_2 = \frac{n}{r} \right\},$$

$\mathcal{A}_3(n, r)$

$$= \left\{ (u_1, u_2, u_3) \in (2\mathbb{Z}_{\geq 0} + 1)^3 : u_1 + u_2 + u_3 = \frac{n}{r} \right\}.$$

$$\tag{5.2}$$

By Lemma 3.9 and Propositions 3.10 and 3.11, our goal is the computation of

$$a_n^p = \sum_{r \mid n} \mu(r) \left[r\alpha_1(n, r) + r\alpha_2(n, r) + \alpha_3(n, r) \right]. \tag{5.3}$$

This, and hence Theorem 1.1, follows from the forthcoming Lemmas 5.3, 5.6, and 5.7.

5.1   A preliminary arithmetical result

The following lemma will be useful in the sequel.

Lemma 5.1. Let $n \geq 1$. Then

$$\sum_{r \mid n} r\mu(r) \sum_{d \mid n/r} \mu(d)\sigma_k\left(\frac{n}{rd}\right) = n^k \sum_{r \mid n} \frac{\mu(r)}{r^{k-1}},$$

$$\tag{5.4}$$

$$\sum_{r \mid n} \mu(r) \sum_{d \mid n/r} \frac{\mu(d)}{d} \sigma_1\left(\frac{n}{rd}\right) = n \sum_{d \mid n} \frac{\mu(d)}{d^2}.$$

$\qquad\square$

Proof. Consider the function

$$f = \left(\mathrm{id}^\ell \, \mu\right) * \mu * \sigma_k. \tag{5.5}$$

Then

$$L(f, s) = \frac{\zeta(s - k)}{\zeta(s - \ell)} = L\left(\mathrm{id}^\ell \, \mu, s\right) L\left(\mathrm{id}^k, s\right), \tag{5.6}$$

hence

$$f = \left(\mathrm{id}^{\ell}\, \mu \right) * \mathrm{id}^{k}. \tag{5.7}$$

The lemma follows by taking $\ell = 1$ for the first equality and $\ell = k = 1$ for the second. ∎

Remark 5.2. Note that

$$\sum_{r \mid n} \frac{\mu(r)}{r^2} = \prod_{p \mid n} \left(1 - \frac{1}{p^2} \right). \tag{5.8}$$

5.2   Two cylinders and odd heights

Here, we compute the sum

$$\sum_{r \mid n} r\mu(r)\alpha_1(n, r). \tag{5.9}$$

More precisely, we prove the following lemma.

Lemma 5.3. The number of type A primitive surfaces with n squares and two cylinders of odd height is

$$\frac{n^2(n-1)}{8} \sum_{r \mid n} \frac{\mu(r)}{r^2}. \tag{5.10}$$
□

Write

$$\alpha_1(n, r) = \gamma_1(n, r) - \widetilde{\alpha}_1(n, r) \tag{5.11}$$

with

$$\gamma_1(n, r) = \sum_{(h_1, u_1, h_2, u_2) \in \mathcal{C}(n, r)} u_1 u_2,$$

$$\widetilde{\alpha}_1(n, r) = \sum_{(h_1, u_1, h_2, u_2) \in \widetilde{\mathcal{A}}_1(n, r)} u_1 u_2, \tag{5.12}$$

where

$$\mathcal{C}(n, r) = \left\{ (h_1, u_1, h_2, u_2) \in \mathbb{Z}_{>0}^4 : (h_1, h_2) = 1,\ u_1 < u_2,\ h_1 u_1 + h_2 u_2 = \frac{n}{r} \right\} \tag{5.13}$$

and (recalling that n is odd)

$$\widetilde{\mathcal{A}}_1(n,r) = \left\{ (h_1, u_1, h_2, u_2) \in \mathbb{Z}_{>0}^4 : \left| \begin{array}{l} (h_1, h_2) = 1, \\ h_1 \text{ or } h_2 \text{ even,} \end{array} \right. u_1 < u_2, \ h_1 u_1 + h_2 u_2 = \frac{n}{r} \right\}. \tag{5.14}$$

Note that the sum

$$\sum_{r|n} r\mu(r)\gamma_1(n,r) \tag{5.15}$$

is the total number of primitive surfaces with two cylinders. Lemma 5.3 is a consequence of the two following Lemmas 5.4 and 5.5 and of (5.11).

5.2.1   Surfaces with two cylinders.   We prove the following result.

Lemma 5.4. For n odd, the number of primitive surfaces with n squares and two cylinders is

$$\frac{n^2(5n-18)}{24} \sum_{r|n} \frac{\mu(r)}{r^2} + \frac{n}{2}\varphi(n), \tag{5.16}$$

where φ is the Euler function. $\square$

Using Möbius inversion formula, one obtains

$$\gamma_1(n,r) = \sum_{d|n/r} \mu(d) \sum_{\substack{(i_1, u_1, i_2, u_2) \in \mathbb{Z}_{>0}^4 \\ u_1 < u_2 \\ i_1 u_1 + i_2 u_2 = n/(rd)}} u_1 u_2$$

$$= \gamma_{1,1}(n,r) - \gamma_{1,2}(n,r) \tag{5.17}$$

with

$$\gamma_{1,1}(n,r) = \frac{1}{2} \sum_{d|n/r} \mu(d) \sum_{\substack{(i_1, u_2, i_2, u_2) \in \mathbb{Z}_{>0}^4 \\ i_1 u_1 + i_2 u_2 = n/(rd)}} u_1 u_2,$$

$$\gamma_{1,2}(n,r) = \frac{1}{2} \sum_{d|n/r} \mu(d) \sum_{\substack{(i_1, i_2, u) \in \mathbb{Z}_{>0}^3 \\ (i_1 + i_2)u = n/(rd)}} u^2. \tag{5.18}$$

One has

$$
\begin{aligned}
\gamma_{1,1}(n, r) &= \frac{1}{2} \sum_{d | n/r} \mu(d) \sum_{\substack{(\nu_1, \nu_2) \in \mathbb{Z}_{>0}^2 \\ \nu_1 + \nu_2 = n/(rd)}} \sum_{w_1 | \nu_1} w_1 \sum_{w_2 | \nu_2} w_2 \\
&= \frac{1}{2} \sum_{d | n/r} \mu(d) S_1 \left(\frac{n}{rd} \right).
\end{aligned}
\tag{5.19}
$$

By Proposition 4.4, this can be linearised to

$$
\begin{aligned}
\gamma_{1,1}(n, r) &= \frac{5}{24} \sum_{d | n/r} \mu(d) \sigma_3 \left(\frac{n}{rd} \right) - \frac{n}{4r} \sum_{d | n/r} \frac{\mu(d)}{d} \sigma_1 \left(\frac{n}{rd} \right) \\
&\quad + \frac{1}{24} \sum_{d | n/r} \mu(d) \sigma_1 \left(\frac{n}{rd} \right)
\end{aligned}
\tag{5.20}
$$

so as to obtain

$$
\sum_{r | n} r \mu(r) \gamma_{1,1}(n, r) = \left(\frac{5}{24} n^3 - \frac{1}{4} n^2 \right) \sum_{r | n} \frac{\mu(r)}{r^2}
\tag{5.21}
$$

thanks to Lemma 5.1.

Next, one has

$$
\gamma_{1,2}(n, r) = \frac{1}{2} \sum_{d | n/r} \mu(d) \sum_{\nu | n/(rd)} \nu^2 \left(\frac{n}{r\, d\nu} - 1 \right)
\tag{5.22}
$$

so that

$$
\begin{aligned}
\sum_{r | n} r \mu(r) \gamma_{1,2}(n, r) &= \frac{n}{2} \sum_{r | n} \mu(r) \sum_{d | n/r} \frac{\mu(d)}{d} \sigma_1 \left(\frac{n}{rd} \right) \\
&\quad - \frac{1}{2} \sum_{r | n} r \mu(r) \sum_{d | n/r} \mu(d) \sigma_2 \left(\frac{n}{rd} \right) \\
&= \frac{n^2}{2} \sum_{r | n} \frac{\mu(r)}{r^2} - \frac{n}{2} \varphi(n)
\end{aligned}
\tag{5.23}
$$

by Lemma 5.1.

Finally, reporting (5.21) and (5.23) in (5.17) leads to Lemma 5.4.

5.2.2   Even product of heights.   Let us now compute the contribution of $\widetilde{\alpha}_1(n, r)$.

Lemma 5.5. The number of type A primitive surfaces with n squares and two cylinders, one having even height, is

$$\frac{n^2(2n-15)}{24} \sum_{r|n} \frac{\mu(r)}{r^2} + \frac{n}{2}\varphi(n). \tag{5.24}$$

$\square$

Write

$$\widetilde{\alpha}_1(n, r) = \widetilde{\alpha}_{1,1}(n, r) - \widetilde{\alpha}_{1,2}(n, r) \tag{5.25}$$

with, recalling again that n is odd,

$$\widetilde{\alpha}_{1,1}(n, r) = \frac{1}{2} \sum_{d|n/r} \mu(d) \sum_{(i_1, u_1, i_2, u_2) \in \widetilde{\mathcal{A}}_{1,1}(n,r)} u_1 u_2,$$

$$\widetilde{\alpha}_{1,2}(n, r) = \frac{1}{2} \sum_{d|n/r} \mu(d) \sum_{(i_1, i_2, u) \in \widetilde{\mathcal{A}}_{1,2}(n,r)} u^2, \tag{5.26}$$

where

$$\widetilde{\mathcal{A}}_{1,1} = \left\{ (i_1, u_1, i_2, u_2) \in \mathbb{Z}_{>0}^4 : i_1 \text{ or } i_2 \text{ even}, i_1 u_1 + i_2 u_2 = \frac{n}{dr} \right\},$$

$$\widetilde{\mathcal{A}}_{1,2} = \left\{ (i_1, i_2, u) \in \mathbb{Z}_{>0}^3 : i_1 \text{ or } i_2 \text{ even}, (i_1 + i_2)u = \frac{n}{dr} \right\}. \tag{5.27}$$

Since i_1 and i_2 are not simultaneously even, one has

$$\widetilde{\alpha}_{1,1}(n, r) = \sum_{d|n/r} \mu(d) \sum_{\substack{(v_1, v_2) \in \mathbb{Z}_{>0}^2 \\ v_1 + v_2 = n/(dr)}} \sum_{\substack{i_1|v_1 \\ i_1 \text{ even}}} \sum_{i_2|v_2} i_2 = \sum_{d|n/r} \mu(d) S_2\left(\frac{n}{dr}\right). \tag{5.28}$$

Using Proposition 4.4 and Lemma 5.1, one obtains

$$\sum_{r|n} r\mu(r)\widetilde{\alpha}_{1,1}(n, r) = \left(\frac{1}{12}n^3 - \frac{1}{8}n^2\right) \sum_{r|n} \frac{\mu(r)}{r^2}. \tag{5.29}$$

Next,

$$\widetilde{\alpha}_{1,2}(n, r) = \frac{1}{2} \sum_{d|n/r} \mu(d) \sum_{u|n/(dr)} u^2\left(\frac{n}{r\,du} - 1\right). \tag{5.30}$$

Lemma 5.1 gives

$$\sum_{r|n} r\mu(r)\widetilde{\alpha}_{1,2}(n,r) = \frac{n^2}{2}\sum_{r|n}\frac{\mu(r)}{r^2} - \frac{n}{2}\varphi(n). \qquad (5.31)$$

Reporting (5.29) and (5.31) in (5.25) leads to Lemma 5.5.

5.3   Two cylinders with even product of heights

Compute at last the sum

$$\sum_{r|n} r\mu(r)\alpha_2(n,r). \qquad (5.32)$$

Lemma 5.6. The number of type A primitive surfaces with n squares and two cylinders, one having an even height, the other having an even length, is

$$\frac{n^2(n-3)}{48}\sum_{r|n}\frac{\mu(r)}{r^2}. \qquad (5.33)$$
$\square$

Since n is odd, one has

$$\alpha_2(n,r) = \frac{1}{4}\sum_{d|n/r}\mu(d)\sum_{(i_1,u_1,i_2,u_2)\in\widehat{\mathcal{A}}_{1,2}(n,r,d)} u_1 u_2 \qquad (5.34)$$

with

$$\widehat{\mathcal{A}}_{1,2}(n,r,d) = \left\{ (i_1,u_1,i_2,u_2)\in\mathbb{Z}_{>0}^4 : \begin{array}{|l} i_1 \text{ and } u_1 \text{ even,} \\ \qquad \text{or} \qquad i_1 u_1 + i_2 u_2 = \dfrac{n}{rd} \\ i_2 \text{ and } u_2 \text{ even,} \end{array} \right\}. \qquad (5.35)$$

Hence,

$$\begin{aligned} \alpha_2(n,r) &= \frac{1}{2}\sum_{d|n/r}\mu(d)\sum_{\substack{(i_1,u_1,i_2,u_2)\in\mathbb{Z}_{>0}^4 \\ i_1 \text{ and } u_1 \text{ even} \\ i_1 u_1 + i_2 u_2 = n/(rd)}} u_1 u_2 \\ &= \sum_{d|n/r}\mu(d)S_4\left(\frac{n}{rd}\right). \end{aligned} \qquad (5.36)$$

The result follows from Proposition 4.4 and Lemma 5.1.

5.4   One cylinder

The counting in that case is more direct.

Lemma 5.7.   The number of type A primitive surfaces with n squares and one cylinder is

$$\frac{n^3}{24} \sum_{r|n} \frac{\mu(r)}{r^2}. \tag{5.37}$$

$\square$

One actually has

$$\sum_{r|n} \mu(r)\alpha_3(n,r) = \frac{n}{3} \sum_{r|n} \mu\left(\frac{n}{r}\right) \#\left\{ (\nu_1, \nu_2, \nu_3) \in \mathbb{Z}_{\geq 0}^3 : \nu_1 + \nu_2 + \nu_3 = \frac{r-3}{2} \right\}$$

$$= \frac{n}{3} \sum_{r|n} \mu\left(\frac{n}{r}\right) \frac{r^2 - 1}{8} \tag{5.38}$$

$$= \frac{1}{24} n^3 \sum_{r|n} \frac{\mu(r)}{r^2}.$$

5.5   Computation of a generating series

The number of non-necessarily primitive surfaces with an odd number n of squares of type A is given by

$$a_n = \sum_{d|n} \sigma_1\left(\frac{n}{d}\right) a_d^p. \tag{5.39}$$

Even though this does not have any geometric sense, one can define numbers a_n^p and a_n by these formulae for even $n \geq 2$. We will compute the Fourier series attached to the resulting sequence $(a_n)_{n \in \mathbb{Z}_{>0}}$. Corollary 1.3 follows directly from the following proposition.

Proposition 5.8.   Let $n \geq 1$. Then

$$a_n = \frac{3}{16}\left[\sigma_3(n) - n\sigma_1(n)\right]. \tag{5.40}$$

$\square$

Proof.   We use the basic facts of Section 3.1. We have

$$a_n = \frac{3}{16}\left[k_3(n) - k_2(n)\right], \tag{5.41}$$

where, for $\ell \in \mathbb{Z}$, the arithmetical function k_ℓ is defined by

$$k_\ell = \sigma_1 * \left(\mathrm{id}^\ell \Psi\right) \tag{5.42}$$

with

$$\Psi(n) = \sum_{r|n} \frac{\mu(r)}{r^2} = \quad * \left(\mu\,\mathrm{id}^{-2}\right)(n). \tag{5.43}$$

We deduce that

$$L\left(k_\ell, s\right) = \frac{\zeta(s)\zeta(s-1)\zeta(s-\ell)}{\zeta(s-\ell+2)}, \tag{5.44}$$

hence

$$k_3 = \sigma_3, \qquad k_2 = \mathrm{id}\,\sigma_1. \tag{5.45}$$

$\blacksquare$

6   The associated Fourier series

Recall that the two weight 2 modular forms Φ_2 and Φ_4 on $\Gamma_0(2)$ and $\Gamma_0(4)$, respectively, have been defined in (4.27). In this section, we prove Theorem 1.5. Since we want to eliminate the coefficients of even order, it is natural to consider the Fourier series obtained by twisting all coefficients by a modulus 2 character. By Proposition 4.7, one obtains a quasimodular form of weight 4, depth less than or equal to 2 on $\Gamma_0(4)$, hence a linear combination of E_4, $E_{2,4}$, $E_{4,4}$, $D\Phi_2$, $D\Phi_4$, and DE_2 (see the proof of Proposition 4.4). The coefficients of this combination are found by computation of the first seven Fourier coefficients.

Acknowledgments

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