Mathematatische Zeitschrift
https://doi.org/10.1007/s00209-018-2191-1

Mathematische Zeitschrift

On Fourier coefficients of modular forms of half integral weight at squarefree integers

Y.-J. Jiang[1] · Y.-K. Lau[2] · G.-S. Lü[3] · E. Royer[4] · J. Wu[5]

Received: 26 February 2016 / Accepted: 14 November 2018

Abstract
We show that the Dirichlet series associated to the Fourier coefficients of a half-integral weight Hecke eigenform at squarefree integers extends analytically to a holomorphic function in the half-plane $\Re e\, s > \frac{1}{2}$. This exhibits a high fluctuation of the coefficients at squarefree integers and improves a sign-change result in Lau et al. (Mathematika 62:866–883, 2016).

Keywords Fourier coefficients · Half-integral weight modular forms · Dirichlet series

Mathematics Subject Classification 11F30

1 Introduction

Some modular forms are endowed with nice arithmetic properties, for which techniques in analytic number theory can be applied to unveil their extraordinary features. For instance, Matomäki and Radziwill [10] made an important progress for multiplicative functions with

✉ Y.-K. Lau
yklau@maths.hku.hk

Y.-J. Jiang
yujiaoj@hotmail.com

G.-S. Lü
gslv@sdu.edu.cn

E. Royer
emmanuel.royer@math.univ-bpclermont.fr

J. Wu
jie.wu@math.cnrs.fr

1 School of Mathematics and Statistics, Shandong University, Weihai, Weihai 264209, Shandong, China

2 Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong, China

3 Department of Mathematics, Shandong University, Jinan 250100, Shandong, China

4 Université Clermont Auvergne, CNRS, LMBP, 63000 Clermont-Ferrand, France

5 CNRS LAMA 8050, Laboratoire d'Analyse et de Mathématiques Appliquées, Université Paris-Est Créteil, 94010 Créteil cedex, France

Published online: 28 November 2018

an application (amongst many) to give a very sharp result on the holomorphic Hecke cusp eigenforms of integral weight. The Hecke eigenforms of half-integral weight is substantially different from the case of integral weight. A simple illustration is the multiplicativity of their Fourier coefficients. If f is a Hecke eigenform of integral weight (for $SL_2(\mathbb{Z})$), its Fourier coefficient $a_f(m)$ will be factorized into $a_f(m) = \prod_{p^r \| m} a_f(p^r)$. However, for a Hecke eigenform $\mathfrak{f}$ of half-integral weight (for $\Gamma_0(4)$), we only have $a_{\mathfrak{f}}(tm^2) = a_{\mathfrak{f}}(t) \prod_{p^r \| m} a_{\mathfrak{f}}(p^{2r})$ for any squarefree t, due to Shimura(Both $a_f(1) = a_{\mathfrak{f}}(1) = 1$ are assumed.). This, on one hand, alludes to the mystery of $\{a_{\mathfrak{f}}(t)\}^{\flat}_{t \geqslant 1}$[1] and, on the other hand, provides an interesting object $\{a_{\mathfrak{f}}(n)\}_{n \geqslant 1}$ whose multiplicativity (is limited to the square factors) has no analogue to the classical number-theoretic functions.

The classical divisor function $\tau(n) := \sum_{d|n} 1$ appears to be Fourier coefficients of some Eisenstein series. In the literature there are investigations on $\{\tau(t)\}^{\flat}_{t \geqslant 1}$ and on the associated Dirichlet series $L(s) := \sum^{\flat}_{t \geqslant 1} \tau(t) t^{-s}$, which is however rather obscure. Using the multiplicative properties of $\tau(n)$, $L(s)$ is connected to the reciprocal of the Riemann zeta-function $\zeta(2s)^3$, and it extends analytically to a holomorphic function on (a slightly bigger region containing) the half-plane $\Re e\, s > \frac{1}{2}$ except at $s = 1$. A further extension is equivalent to a progress towards the Riemann Hypothesis.

On the other hand, to study the sign-changes in $\{a_{\mathfrak{f}}(t)\}^{\flat}_{t \geqslant 1}$, Hulse et al. [3] recently considered $L^{\flat}_{\mathfrak{f}}(s) := \sum^{\flat}_{t \geqslant 1} \lambda_{\mathfrak{f}}(t) t^{-s}$ (where $\lambda_{\mathfrak{f}}(t) = a_{\mathfrak{f}}(t) t^{-(\ell/2-1/4)}$). Interestingly they showed that $L^{\flat}_{\mathfrak{f}}(s)$ extends analytically to a holomorphic function in $\Re e\, s > \frac{3}{4}$. One naturally asks how far $L^{\flat}_{\mathfrak{f}}(s)$ can further extend to. Compared with the case of $\tau(n)$ but without adequate multiplicativity, a continuation to the region $\Re e\, s > \frac{1}{2}$ is curious, non-trivial and plausibly (very close to) the best attainable with current technology.

The argument of proof in [3] is based on the convexity principle and includes two key ingredients:

(a) the inequality $\lambda_{\mathfrak{f}}(tr^2) \ll_{\varepsilon} |\lambda_{\mathfrak{f}}(t)| r^{\varepsilon}$,
(b) the functional equations of the twisted L-functions for $\mathfrak{f}$ by additive characters $\mathrm{e}(un/d)$.

The inequality (a) is a substitute for the unsettled Ramanujan Conjecture for half-integral weight Hecke eigenforms, and this is derived from the Shimura correspondence and the Deligne bound for modular forms of integral weight. According to various d's, the functional equations of (b) involves the Fourier expansions of $\mathfrak{f}$ at different cusps, which is detailedly computed in [3]. However, due to the multiplier system, the Fourier expansion at the cusp $\frac{1}{2}$ is not of period 1, of which Hulse et al seemed not aware. We shall propose an amendment in Sect. 4.

Our main goal is to prove that $L^{\flat}_{\mathfrak{f}}(s)$ extends analytically to $\Re e\, s > \frac{1}{2}$. We shall not use the convexity principle but apply the approximate functional equation with the point s close to the line $\Re e\, s = \frac{1}{2}$ (from right). The cancellation amongst the exponential factors and real quadratic characters arising from the twisted L-functions are explored. Without a known Ramanujan Conjecture, the inequality (a) is crucial and indeed we need more—an inequality of the same type for the Fourier coefficients at all cusps, which is done in Sect. 3. There we study the Fourier coefficients of a (complete) Hecke eigenform at the two cusps 0 and $\frac{1}{2}$, and derive some inequalities and bounds useful for analytic approaches, which are of their own interest.

[1] The superscript $^{\flat}$ is to indicate that the index is supported at squarefree integers.

2 Main results

Let $\ell \geqslant 2$ be a positive integer, and denote by $\mathfrak{S}_{\ell+1/2}$ the set of all holomorphic cusp forms of weight $\ell + 1/2$ for the congruence subgroup $\Gamma_0(4)$. The Fourier expansion of $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ at ∞ is written as

$$\mathfrak{f}(z) = \sum_{n \geqslant 1} \lambda_{\mathfrak{f}}(n) n^{\ell/2-1/4}\, \mathrm{e}(nz) \qquad (z \in \mathscr{H}), \tag{2.1}$$

where $\mathrm{e}(z) = \mathrm{e}^{2\pi \mathrm{i} z}$ and $\mathscr{H}$ is the Poincaré upper half plane. Define

$$L_{\mathfrak{f}}^{\flat}(s) := \sum_{t \geqslant 1}^{\flat} \lambda_{\mathfrak{f}}(t) t^{-s} \tag{2.2}$$

for $s = \sigma + \mathrm{i}\tau$ with $\sigma > 1$, where $\sum_{t \geqslant 1}^{\flat}$ ranges over squarefree integers $t \geqslant 1$.

Theorem 1 *Let $\ell \geqslant 2$ be a positive integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be a complete Hecke eigenform. The series $L_{\mathfrak{f}}^{\flat}(s)$ in (2.2) extends analytically to a holomorphic function on $\Re e\, s > \frac{1}{2}$. Moreover, for any $\varepsilon > 0$ we have*

$$L_{\mathfrak{f}}^{\flat}(s) \ll_{\mathfrak{f},\varepsilon} (|\tau| + 1)^{1-\sigma+2\varepsilon} \qquad \left(\tfrac{1}{2} + \varepsilon \leqslant \sigma \leqslant 1 + \varepsilon, \tau \in \mathbb{R}\right), \tag{2.3}$$

where the implied constant depends on $\mathfrak{f}$ and ε only.

Remark 1 It follows immediately the Riesz mean $\sum_{t \leqslant x}^{\flat} (1 - t/x)\lambda_{\mathfrak{f}}(t) \ll x^{1/2+\varepsilon}$, exhibiting a support towards square-root cancellation of $\{\lambda_{\mathfrak{f}}(t)\}_{n \geqslant 1}^{\flat}$.

An application of Theorem 1 is a better lower bound (than [9, Theorem 4]) for the sign-changes of $\{\lambda_{\mathfrak{f}}(t)\}_{t \geqslant 1}^{\flat}$ with $t \in [1, x]$. Together with a refinement on the mean square formula, we shall prove the next theorem in Sect. 8.

Theorem 2 *Let $\ell \geqslant 2$ be an integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ a complete Hecke eigenform such that its Fourier coefficients are real. Denote by $\mathfrak{C}_{\mathfrak{f}}^{\flat}(x)$ the number of sign changes of $\lambda_{\mathfrak{f}}(t)$ where t ranges over squarefree integers in $[1, x]$.*[2] *Let $\varrho \in (0, \frac{1}{4})$ be defined as in (3.13), and ϑ any number satisfying*

$$0 < \vartheta < \min\left(\tfrac{1-2\varrho}{3}, \tfrac{1}{4}\right).$$

Then for all $x \geqslant x_0(\mathfrak{f}, \vartheta)$,

$$\mathfrak{C}_{\mathfrak{f}}^{\flat}(x) \gg_{\mathfrak{f},\vartheta} x^{\vartheta}$$

where the constant $x_0(\mathfrak{f}, \vartheta)$ and the implied constant depend on $\mathfrak{f}$ and ϑ only.

3 Half-integral weight cusp forms for $\Gamma_0(4)$

We follow Shimura [14] to explicate the definition of $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$. The main aim is to discuss some properties of the Fourier coefficients at *all* cusps when $\mathfrak{f}$ is a complete Hecke eigenform.

Let $GL_2^+(\mathbb{R})$ be the set of all real 2×2 matrices with positive determinant. Define $\widetilde{G}$ to be the set of all $(\alpha, \varphi(z))$ where $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and $\varphi(z)$ is a holomorphic function

[2] See [9] for explanation in detail.

on $\mathscr{H}$ such that

$$\varphi(z)^2 := \varsigma \det(\alpha)^{-1/2}(cz+d), \quad \text{for some } \varsigma \in \mathbb{C} \text{ with } |\varsigma| = 1.$$

Then $\widetilde{G}$ is a group under the composition law $(\alpha, \varphi(z))(\beta, \psi(z)) = (\alpha\beta, \varphi(\beta z)\psi(z))$. The projection map $(\alpha, \varphi(z)) \mapsto \alpha$ is a surjective homomorphism from $\widetilde{G}$ to $GL_2^+(\mathbb{R})$. We write $(\alpha, \varphi(z))_* = \alpha$. Let f be any complex-valued function on $\mathscr{H}$. The slash operator $\xi \mapsto f|_{[\xi]}$, defined as

$$f|_{[\xi]} := \varphi(z)^{-(2\ell+1)} f(\alpha z) \quad \text{if } \xi = (\alpha, \varphi(z)),$$

gives an anti-homomorphism on $\widetilde{G}$, i.e. $f|_{[\xi\eta]} = (f|_{[\xi]})|_{[\eta]}$.

Define for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ and $z \in \mathscr{H}$,

$$j(\gamma, z) := \frac{\theta(\gamma z)}{\theta(z)} = \varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{1/2},$$

where $\varepsilon_d = 1$ or i according as $d \equiv 1$ or $3 \pmod 4$, the extended Jacobi symbol $\left(\frac{c}{d}\right)$ and the square root $(cz+d)^{1/2}$ are defined as in [14]. The map $\gamma \mapsto \gamma^*$ with $\gamma^* := (\gamma, j(\gamma, z))$ is an one-to-one homomorphism from $\Gamma_0(4)$ to $\widetilde{G}$. For $\gamma \in \Gamma_0(4)$, we will abbreviate $f|_{[\gamma^*]}$ as $f|_{[\gamma]}$.

A cusp form $\mathfrak{f}$ of weight $\ell + 1/2$ for $\Gamma_0(4)$ is a holomorphic function on $\mathscr{H}$ such that

1° $\mathfrak{f}|_{[\gamma]} = \mathfrak{f}$ for all $\gamma \in \Gamma_0(4)$,
2° $\mathfrak{f}$ admits a Fourier series expansion at every cusp $\mathfrak{a} \in \{0, -\frac{1}{2}, \infty\}$,

$$\mathfrak{f}|_{[\rho]} = \sum_{\substack{n\in\mathbb{Z} \\ n+r>0}} c_n \,\mathrm{e}((n+r)z).$$

Here $\rho \in \widetilde{G}$ satisfies that its projection is a scaling matrix for the cusp $\mathfrak{a}$, i.e. $\rho_*(\infty) = \mathfrak{a}$, and for some $|t| = 1$,

$$\rho^{-1}\eta^*\rho = (\eta_\infty, t) \quad \text{with } \eta_\infty := \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix},$$

where η is a generator of the stabilizer $\Gamma_{\mathfrak{a}}$ in $\Gamma_0(4)$ for the cusp $\mathfrak{a}$. The value of $r \in [0, 1)$ is determined by $\mathrm{e}(r) = t^{2\ell+1}$. (See [14, p. 444].)

3.1 Fourier expansions at the three cusps

Explicitly we take $\rho = \rho_{\mathfrak{a}}$ where

$$\rho_{\mathfrak{a}} = \begin{cases} \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \ 1\right) & \text{for } \mathfrak{a} = \infty, \\ \left(\begin{pmatrix} 1 & \\ -2 & 1 \end{pmatrix}, \ (-2z+1)^{1/2}\right) & \text{for } \mathfrak{a} = -\frac{1}{2}, \\ \left(\begin{pmatrix} & -1 \\ 4 & \end{pmatrix}, \ 2^{1/2}(-\mathrm{i}z)^{1/2}\right) & \text{for } \mathfrak{a} = 0. \end{cases} \tag{3.1}$$

Set $\eta_{\mathfrak{a}} = \rho_{\mathfrak{a}*}\eta_\infty\rho_{\mathfrak{a}*}^{-1}$. Then $\eta_{\mathfrak{a}} \in \Gamma_0(4)$ for all the three cusps. A direct checking shows that $\rho_{\mathfrak{a}}^{-1}\eta_{\mathfrak{a}}^*\rho_{\mathfrak{a}} = (\eta_\infty, t_{\mathfrak{a}})$ where $t_{\mathfrak{a}} = 1, \mathrm{i}, 1$ for $\mathfrak{a} = \infty, -\frac{1}{2}, 0$, respectively. (When $\mathfrak{a} = -\frac{1}{2}$, the factor $\varepsilon_{-1}^{-1}\left(\frac{-4}{-1}\right)$ inside $j(\eta_{\mathfrak{a}}^{*}, z)$ equals i.) Hence, for $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$, $\mathfrak{f}(z+1) = \mathfrak{f}(z)$ (note $\mathfrak{f}|_{[\rho_\infty]} = \mathfrak{f}$) and $\mathfrak{f}|_{[\rho_0]}(z+1) = \mathfrak{f}|_{[\rho_0]}(z)$, while for $\mathfrak{a} = -\frac{1}{2}$,

$$\mathfrak{f}|_{[\rho_{\mathfrak{a}}]}(z+1) = t_{-1/2}^{2\ell+1}\mathfrak{f}|_{[\rho_{\mathfrak{a}}\eta_\infty]}(z) = \mathrm{i}^{2\ell+1}\big(\mathfrak{f}|_{[\eta_{\mathfrak{a}}^{*}]}\big)|_{[\rho_{\mathfrak{a}}]}(z) = \mathrm{i}^{2\ell+1}\mathfrak{f}|_{[\rho_{\mathfrak{a}}]}(z). \tag{3.2}$$

Let $\alpha = \left(\begin{pmatrix} 4 & \\ & 1\end{pmatrix}, 2^{-1/2}\right)$. For our purpose, we set

$$\mathfrak{g}(z) := \big(\mathfrak{f}|_{[\rho_{-1/2}]}\big)|_{[\alpha]}(z) = 2^{\ell+1/2}\mathfrak{f}|_{[\rho_{-1/2}]}(4z) \quad \text{and} \quad \mathfrak{h}(z) := \mathfrak{f}|_{[\rho_0]}(z). \tag{3.3}$$

Their Fourier series expansions (at ∞) are of the form

$$\begin{aligned}\mathfrak{g}(z) &= 2^{\ell+1/2}\sum_{n\geqslant 0} c_n\, \mathrm{e}\left((4n + (2 + (-1)^{\ell-1}))z\right)\\ &= 2^{\ell+1/2}\sum_{n\geqslant 1}\lambda_{\mathfrak{g}}(n) n^{\ell/2-1/4}\,\mathrm{e}(nz), \quad \text{say},\end{aligned} \tag{3.4}$$

where the sequence $\{\lambda_{\mathfrak{g}}(n)\}$ is supported on positive integers $n \equiv (-1)^{\ell} \pmod 4$, and

$$\mathfrak{h}(z) = \sum_{n\geqslant 1}\lambda_{\mathfrak{h}}(n) n^{\ell/2-1/4}\,\mathrm{e}(nz). \tag{3.5}$$

Remark 2 (i) The cusp form $\mathfrak{h}(z)$ is $\mathfrak{f}_0(z)$ in [3] but $\mathfrak{g}(z) = \mathfrak{f}_{\frac{1}{2}}(4z)$, not $\mathfrak{f}_{\frac{1}{2}}(z)$, there. The Fourier expansion of $\mathfrak{f}_{\frac{1}{2}}(z)$ at ∞ is of the form $\sum_{n\geqslant 1} c_n\,\mathrm{e}((n+\frac{1}{4})z)$.
(ii) The form $\mathfrak{h}$ is a cusp form for $\Gamma_0(4)$ but $\mathfrak{g}$ is a cusp form for $\Gamma_0(16)$.
(iii) Using the Rankin–Selberg theory, one can prove that

$$\sum_{n\leqslant x}|\lambda_f(n)|^2 \sim c_f x \qquad (f = \mathfrak{f}, \mathfrak{g} \text{ or } \mathfrak{h}) \tag{3.6}$$

for some constant $c_f > 0$. See [9, Sect. 3], for example. (There the assumption that $\mathfrak{f}$ is a complete Hecke eigenform is not necessary, which is clearly seen from the proof.)

3.2 Eigenform properties of a complete Hecke eigenform at various cusps

Let N be a positive integer divisible by 4, and $p \nmid N$ be any prime. The action of the Hecke operator $\mathsf{T}(p^2)$ on a modular form f of half-integral weight $\ell + 1/2$ for $\Gamma_0(N)$ is defined as (cf. [14, p. 451])

$$\mathsf{T}(p^2)f := p^{\ell-3/2}\left\{\sum_{0\leqslant b<p^2} f|_{[\alpha_b^\star]} + \sum_{1\leqslant h<p} f|_{[\beta_h^\star]} + f|_{[\sigma^\star]}\right\},$$

where

$$\begin{aligned}\alpha_b^\star &:= (\alpha_b, p^{1/2}) = \left(\begin{pmatrix} 1 & b\\ & p^2\end{pmatrix}, p^{1/2}\right),\\ \beta_h^\star &:= \left(\beta_h, \varepsilon_p^{-1}\left(\tfrac{-h}{p}\right)\right) = \left(\begin{pmatrix} p & h\\ & p\end{pmatrix}, \varepsilon_p^{-1}\left(\frac{-h}{p}\right)\right),\\ \sigma^\star &:= (\sigma, p^{-1/2}) = \left(\begin{pmatrix} p^2 & \\ & 1\end{pmatrix}, p^{-1/2}\right).\end{aligned}$$

Suppose $\mathfrak{f}$ is a complete Hecke eigenform, i.e. $\mathsf{T}(p^2)\mathfrak{f} = \omega_p \mathfrak{f}$ for all prime p. One may wonder whether $\mathfrak{g}$ and $\mathfrak{h}$ defined as in (3.3) are eigenforms. We can prove the following.

Lemma 3.1 *Let p be any odd prime. If $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ is an eigenform of $\mathsf{T}(p^2)$, then so are the forms $\mathfrak{g}$ and $\mathfrak{h}$ defined in (3.3) and both have the same eigenvalues as $\mathfrak{f}$.*

Proof Let $N = 4$ or 16, and $\Delta_0 = \Gamma_0(N)^*$ be the image of $\Gamma_0(N)$ under the lifting map. It suffices to show that for (i) $\rho_{-1/2}\alpha$, $N = 16$ and (ii) $\rho = \rho_0$, $N = 4$, the elements $\rho\alpha_b^\star, \rho\beta_h^\star, \rho\sigma^\star$ $(0 \leqslant b < p^2, 1 \leqslant h < p)$ form a set of representatives for

$$\Delta_0 \Big\backslash \left(\Delta_0\sigma^\star\rho \sqcup \bigsqcup_{1 \leqslant h < p^2} \Delta_0\beta_b^\star\rho \sqcup \bigsqcup_{0 \leqslant b < p^2} \Delta_0\alpha_b^\star\rho \right).$$

(i) For the case $\rho = \rho_{-1/2}\alpha$, we check by routine calculation that

$$\begin{aligned}
\Delta_0\rho\sigma^\star &= \Delta_0\alpha^\star_{(p^2+1)/2}\rho, \\
\Delta_0\rho\alpha^\star_{(p^2-1)/8} &= \Delta_0\sigma^\star\rho, \\
\{\Delta_0\rho\beta_h^\star \,:\, 1 \leqslant h < p\} &= \{\Delta_0\alpha_d^\star\rho \,:\, p \parallel (1-2d)\}, \\
\{\Delta_0\rho\alpha_b^\star \,:\, p \nmid (1+8b)\} &= \{\Delta_0\alpha_d^\star\rho \,:\, p \nmid (1-2d)\}, \\
\{\Delta_0\rho\alpha_d^\star \,:\, p \parallel (1+8b)\} &= \{\Delta_0\beta_h^\star\rho \,:\, 1 \leqslant h < p\}.
\end{aligned}$$

For example, from

$$\rho = \rho_{-1/2}\alpha = \left(\begin{pmatrix} 4 & \\ -8 & 1 \end{pmatrix}, 2^{-1/2}(-8z+1)^{1/2} \right)$$

we obtain $\rho_*\beta_h\rho_*^{-1} = \gamma\alpha_d$, where

$$\gamma = \begin{pmatrix} p+8h & (4h - d(p+8h))p^{-2} \\ -16h & (p-8h+16hd)p^{-2} \end{pmatrix} \in \Gamma_0(16)$$

if we take $1 \leqslant d < p^2$ such that $d(p+8h) \equiv 4h \pmod{p^2}$. Note that this choice implies $p - 8h + 16hd \equiv p(1-2d) \pmod{p^2}$ and $dp \equiv 4h(1-2d) \pmod{p^2}$. The latter implies $p \mid (1-2d)$, so the former is $\equiv 0 \pmod{p^2}$. Next the φ-part of $\rho\beta_h^\star\rho^{-1}$ is

$$\varepsilon_p^{-1} \left(\frac{-h}{p} \right) (-16hz + p - 8h)^{1/2} p^{-1/2}.$$

To evaluate the φ-part of $\gamma^*\alpha_d^\star$, we remark that $j(\gamma, z) = j(\gamma^{-1}, \gamma z)^{-1}$ and thus consider γ^{*-1} whose j-part is simply

$$\varepsilon_{p+8h}^{-1} \left(\frac{16h}{p+8h} \right) (16hz + p + 8h)^{1/2}.$$

Hence, the φ-part of $\gamma^*\alpha_d^\star$ is

$$\varepsilon_p \left(\frac{h}{p} \right) (-16\gamma\alpha_d z + p + 8h)^{-1/2} p^{1/2}.$$

From $\gamma\alpha_d = \rho_*\beta_h\rho_*^{-1}$ and $\varepsilon_p\left(\frac{-1}{p}\right) = \varepsilon_p^{-1}$, we easily verify this case. The other cases are checked in the same way.

(ii) For the case $\rho = \rho_0$, we find similarly that

$$\begin{aligned}
\Delta_0\rho\sigma^\star &= \Delta_0\alpha_0^\star\rho, \\
\Delta_0\rho\alpha_0^\star &= \Delta_0\sigma^\star\rho, \\
\{\Delta_0\rho\beta_h^\star \,:\, 1 \leqslant h < p\} &= \{\Delta_0\alpha_{pd}^\star\rho \,:\, 1 \leqslant d < p\}, \\
\{\Delta_0\rho\alpha_d^\star \,:\, p \parallel b\} &= \{\Delta_0\beta_h^\star\rho \,:\, 1 \leqslant h < p\}, \\
\{\Delta_0\rho\alpha_b^\star \,:\, p \nmid b\} &= \{\Delta_0\alpha_d^\star\rho \,:\, p \nmid d\}.
\end{aligned}$$

□

3.3 Shimura's correspondence and bounding coefficients

Let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$, not necessarily a complete Hecke eigenform. By Shimura's theory [14, Sect. 3], for any squarefree $t \geqslant 1$, there is a cusp form $\mathrm{Sh}_t\mathfrak{f}$ of weight 2ℓ for $\Gamma_0(2)$ such that

$$t^{\ell/2-1/4}L(s+\tfrac{1}{2},\chi_t)\sum_{n\geqslant 1}\lambda_{\mathfrak{f}}(tn^2)n^{-s} = L(s,\mathrm{Sh}_t\mathfrak{f}), \tag{3.7}$$

where $L(\cdot,\chi_t)$ is the Dirichlet L-function associated to the character

$$\chi_t(n) = \chi_0(n)\left(\frac{-1}{n}\right)^{\ell}\left(\frac{t}{n}\right)$$

(χ_0 is the principal character mod 4) and $L(s,F) := \sum_{n\geqslant 1}\lambda_F(n)n^{-s}$ is the L-function for the cusp form of integral weight 2ℓ with nebentypus χ_0^2,

$$F(z) = \sum_{n\geqslant 1}\lambda_F(n)n^{(2\ell-1)/2}\,\mathrm{e}(nz).$$

The Shimura lift $\mathfrak{f} \mapsto \mathrm{Sh}_t\mathfrak{f}$ commutes with Hecke operators: $\mathrm{Sh}_t(\mathsf{T}(p^2)\mathfrak{f}) = T(p)(\mathrm{Sh}_t\mathfrak{f})$ for all primes p.[3] It follows that the coefficients $\lambda_{\mathfrak{f}}(mp^{2r})$ satisfy a recurrence relation in r when $\mathfrak{f}$ is a $\mathsf{T}(p^2)$-Hecke eigenform. Moreover, if $\mathfrak{f}$ is a Hecke eigenform of $\mathsf{T}(p^2)$ for all $p \notin \mathcal{S}$ (where $\mathcal{S}$ is any set of primes), the right-hand side of (3.7) will admit a factorization (see Corollary 1.8 and Main Theorem in [14])

$$t^{\ell/2-1/4}L(s+\tfrac{1}{2},\chi_t)\sum_{n\geqslant 1}\frac{\lambda_{\mathfrak{f}}(tn^2)}{n^s} = \sum_{\substack{n\geqslant 1\\ p|n\Rightarrow p\in\mathcal{S}}}\frac{\lambda_{\mathrm{Sh}_t\mathfrak{f}}(n)}{n^s}\prod_{p\notin\mathcal{S}}\left(1-\frac{\omega_p}{p^s}+\frac{\chi_0(p)}{p^{2s}}\right)^{-1}, \tag{3.8}$$

where $\mathsf{T}(p^2)\mathfrak{f} = \omega_p p^{(2\ell-1)/2}\mathfrak{f}$. Remark that the product $\prod_{p\notin\mathcal{S}}$ remains the same for lifts of different squarefree t's.

The commutativity between Sh_t and $\mathsf{T}(p^2)$ implies that ω_p is also an eigenvalue of the Hecke operator $T(p)$ for $\mathrm{Sh}_t\mathfrak{f}$. Decompose

$$\mathrm{Sh}_t\mathfrak{f}(z) = \sum_i c_i f_i(\ell_i z) \tag{3.9}$$

where each f_i is a newform (of perhaps lower level) and $f_i(\ell_i z)$'s are linearly independent. Let $\mathcal{S}'$ be the set of all prime p dividing the level of $\mathrm{Sh}_t\mathfrak{f}$, so $\mathcal{S}' = \{2\}$ in our case. If $p \notin \mathcal{S}'$,

[3] This commutativity is pointed out in [12, Corollary 3.16] under the extra condition $p \nmid 4tN$, which is relaxed to all primes p in [13].

then $T(p)(f_i(\ell_i z)) = (T(p)f_i)(\ell_i z)$, $\forall\, i$. (See [5, (2.14)] and [6, Sect. 14.7].) Applying $T(p)$ on both sides of (3.9), we thus see that ω_p is the $T(p)$-eigenvalue of some newform (for $p \nmid$ level of $\mathrm{Sh}_t\mathfrak{f}$) and hence

$$|\omega_p| \leqslant 2 \quad \forall p \notin \mathcal{S} \cup \mathcal{S}', \tag{3.10}$$

by Deligne's bound. Consequently we have the following estimate for $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$.

Lemma 3.2 *Let $\mathcal{Q}$ be a (not necessarily finite) set of primes with $2 \in \mathcal{Q}$. Suppose $\mathfrak{f}$ is an Hecke eigenform of $\mathsf{T}(p^2)$ for all $p \notin \mathcal{Q}$. Let $m \geqslant 1$ be any integer decomposed into $m = qr^2$ such that $p \mid r$ implies $p \notin \mathcal{Q}$, and $p^2 \mid q$ implies $p \in \mathcal{Q}$. Then we have*

$$|\lambda_{\mathfrak{f}}(m)| \leqslant |\lambda_{\mathfrak{f}}(q)|\tau(r)^2.$$

Remark 3 Every integer $m \geqslant 1$ decomposes uniquely into the desired form: Decompose m uniquely into $m = tn^2$ where t is squarefree, write $n = ur$ such that $p \mid u$ implies $p \in \mathcal{Q}$ and $p \mid r$ implies $p \notin \mathcal{Q}$, and then set $q = tu^2$.

Proof Let $m = qr^2 = tu^2r^2$ be decomposed as in Remark 3. By (3.8), we see that

$$t^{\ell/2-1/4}\lambda_{\mathfrak{f}}(tu^2r^2) = \left(\sum_{ab=u} \lambda_{\mathrm{Sh}_t\mathfrak{f}}(a)\mu(b)\frac{\chi_t(b)}{\sqrt{b}}\right)\left(\sum_{cd=r} \omega_c\mu(d)\frac{\chi_t(d)}{\sqrt{d}}\right),$$

where ω_c is the coefficient of c^{-s} in $\prod_{p\notin\mathcal{Q}}\left(1 - \omega_p p^{-s} + \chi_0(p)p^{-2s}\right)^{-1}$ and $\mu(d)$ is the Möbius function. The case $r = 1$ tells that the first bracket is $t^{\ell/2-1/4}\lambda_{\mathfrak{f}}(tu^2)$, i.e. $t^{\ell-1/4}\lambda_{\mathfrak{f}}(q)$. Next, since $|\omega_c| \leqslant \tau(c)$ (by (3.10) and its definition), the absolute value of the second bracket is $\leqslant \tau(r)^2$. □

3.4 Bounds for coefficients of a complete Hecke eigenform at all cusps

In case $\mathfrak{f}$ is a complete Hecke eigenform,[4] we may express (3.8) as

$$t^{\ell/2-1/4}L\left(s+\tfrac{1}{2}, \chi_t\right)\sum_{n\geqslant 1}\lambda_{\mathfrak{f}}(tn^2)n^{-s} = t^{\ell/2-1/4}\lambda_{\mathfrak{f}}(t)L(s, F), \tag{3.11}$$

where the Shimiura lift F is a cusp form independent of t. As the Ramanujan's conjecture holds for holomorphic newforms of integral weight, the rth Fourier coefficients of F are $\ll_{\mathfrak{f}} \tau(r)r^{\ell-1/2}$, where the implied constant is independent of t. Consequently the question of the size of $\lambda_{\mathfrak{f}}(m)$ is reduced to the size at the squarefree part of m:

$$\lambda_{\mathfrak{f}}(m) \ll_{\mathfrak{f}} |\lambda_{\mathfrak{f}}(t)|\tau(r)^2 \tag{3.12}$$

if $m = tr^2$ and squarefree t. Due to Iwaniec [4] or Conrey & Iwaniec [1], etc, there are good estimates for

$$\lambda_{\mathfrak{f}}(t) \ll_{\mathfrak{f},\varrho} t^{\varrho} \qquad \forall \text{ squarefree } t, \tag{3.13}$$

for some $0 < \varrho < \frac{1}{4}$. The value of ϱ is $\frac{1}{6} + \varepsilon$ by [1].

We know from Lemma 3.1 that for a complete Hecke eigenform $\mathfrak{f}$, the forms $\mathfrak{g}$ and $\mathfrak{h}$ are eigenforms of $\mathsf{T}(p^2)$ with the same corresponding eigenvalue for all odd prime p. But for $p = 2$, we do not get the same conclusion. This may result in an unpleasant situation

[4] The content of this subsection is not used in the remaining part of the paper but we would include here for its own interest and for applications in other occasions.

of without (3.12). Note that Lemma 3.2 gives at most a bound of the form $|\lambda_f(t2^{2j})|\tau(r)^2$ (where $f = \mathfrak{g}, \mathfrak{h}$). Now we attempt to clarify as much as possible.

In view of [14, Proposition 1.5], the Hecke operator $\mathsf{T}(2^2)$ is the same as the operator U_4 whose action is

$$(f|U_4)(z) = \frac{1}{4}\sum_{\nu \,(\mathrm{mod}\, 4)} f\left(\frac{z+\nu}{4}\right) = \sum_{n\geqslant 1} a(4n)\,\mathrm{e}(nz) \tag{3.14}$$

if $f(z) = \sum_{n\geqslant 1} a(n)\,\mathrm{e}(nz)$. Then it follows easily that $\mathfrak{g}|\mathsf{T}(2^2) = 0$, because by (3.14) and (3.2),

$$(\mathfrak{g}|U_4)(z) = 2^{\ell-3/2}\sum_{\nu \,(\mathrm{mod}\, 4)} \mathfrak{f}|_{[\rho_{-1/2}]}(z+\nu) = 2^{\ell-3/2}\mathfrak{f}|_{[\rho_{-1/2}]}(z)\sum_{0\leqslant\nu\leqslant 3} \mathrm{i}^{\nu(2\ell+1)} \tag{3.15}$$

where the sum is obviously zero. Thus $\mathfrak{g}$ is also a complete Hecke eigenform although it takes the different eigenvalue 0 for $\mathsf{T}(2^2)$, implying the validity (3.12) for $\mathfrak{g}$ as well.

However for the case of $\mathfrak{h}$, we cannot get the conclusion of $\mathsf{T}(2^2)$-eigenform and we shall get the analogous bound via some bypass. To its end, let us recall Niwa's result in [11], cf. Kohnen [7, p. 250], saying that U_4W_4 is Hermitian operator on $\mathfrak{S}_{\ell+1/2}$ and

$$U_4W_4U_4W_4 - \mu U_4W_4 - 2\mu^2 = 0, \tag{3.16}$$

where $\mu = \left(\frac{2}{2\ell+1}\right)2^{\ell-1} = (-1)^{\ell(\ell+1)/2}2^{\ell-1}$ and

$$(f|W_4)(z) = (-2\mathrm{i}z)^{-(\ell+1/2)}f(-\tfrac{1}{4z}) = f|_{[\rho_0]}(z).$$

Suppose $\mathfrak{f}|\mathsf{T}(2^2) = c\mathfrak{f}$ for some scalar c.[5] By (3.16) and $U_4 = \mathsf{T}(2^2)$, we get

$$c(\mathfrak{f}|W_4U_4W_4) - c\mu(\mathfrak{f}|W_4) - 2\mu^2\mathfrak{f} = 0.$$

(Note that the operator acts on $\mathfrak{f}$ from right.) In particular we observe that $c \neq 0$, because otherwise, $-2\mu^2\mathfrak{f} = 0$ implying $\mathfrak{f} = 0$. As $\mathfrak{h} = \mathfrak{f}|W_4$ and W_4 is an involution (i.e. W_4^2 is the identity), we deduce that

$$(\mathfrak{h}|U_4) = \mu\mathfrak{f} + 2\mu^2c^{-1}\mathfrak{h}. \tag{3.17}$$

We separate into two cases:

- Case 1: $c^2 \neq 2\mu^2$.
  We set $\alpha := c\mu/(2\mu^2 - c^2)$ and consider the form $\mathfrak{H} := \mathfrak{h} + \alpha\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$. Then $c\alpha + \mu = 2\alpha\mu^2/c$ and thus by (3.17),

  $$\mathfrak{H}|\mathsf{T}(2^2) = \mathfrak{H}|U_4 = 2\mu^2c^{-1}\mathfrak{H}.$$

  i.e. The cusp form $\mathfrak{H}$ is an eigenform of $\mathsf{T}(2^2)$, and by Lemma 3.1, $\mathfrak{H}$ is also an eigenform of $\mathsf{T}(p^2)$ for all odd primes p. (Note that $\mathfrak{f}$ and $\mathfrak{h}$ have the same $\mathsf{T}(p^2)$-eigenvalue.) Consequently, both coefficients $\lambda_{\mathfrak{H}}(m)$ and $\lambda_{\mathfrak{f}}(m)$ satisfy (3.12). As $\lambda_{\mathfrak{h}}(m) = \lambda_{\mathfrak{H}}(m) - \alpha\lambda_{\mathfrak{f}}(m)$, we establish (3.18) for $\mathfrak{h}$.
- Case 2: $c^2 = 2\mu^2$.
  We infer from (3.17) and (3.14) that for all integers $n \geqslant 1$,

  $$4^{\ell/2-1/4}\lambda_{\mathfrak{h}}(4n) = \mu\lambda_{\mathfrak{f}}(n) + c\lambda_{\mathfrak{h}}(n).$$

[5] Here we write $\mathfrak{f}|\mathsf{T}(p^2)$ for $\mathsf{T}(p^2)\mathfrak{f}$.

Let $d = c/4^{\ell/2-1/4}$. This recurrence relation gives

$$\lambda_{\mathfrak{h}}(4^J n) = d^J \lambda_{\mathfrak{h}}(n) + \frac{\mu}{4^{\ell/2-1/4}} \sum_{1 \leqslant j < J} d^j \lambda_{\mathfrak{f}}\left(4^{J-j} n\right).$$

Note $\mu^2 = 2^{2\ell-2}$, so $|d| = 1$, and (3.12) holds for $\lambda_{\mathfrak{f}}(4^{J-j}n)$. Hence, for any integer $m = tr^2 4^J$ where t is squarefree and r is odd,

$$\lambda_{\mathfrak{h}}(m) \ll |\lambda_{\mathfrak{h}}\left(tr^2\right)| + J^2 |\lambda_{\mathfrak{f}}(t)| \tau(r)^2.$$

By Lemma 3.2 with $\mathcal{Q} = \{2\}$, we get $|\lambda_{\mathfrak{h}}(tr^2)| \ll |\lambda_{\mathfrak{h}}(t)|\tau(r)^2$ and consequently

$$\lambda_{\mathfrak{h}}(m) \ll (|\lambda_{\mathfrak{h}}(t)| + |\lambda_{\mathfrak{f}}(t)|) J^2 \tau(r)^2 \ll (|\lambda_{\mathfrak{h}}(t)| + |\lambda_{\mathfrak{f}}(t)|)\tau\left(r2^J\right)^2.$$

In summary, we have proved the following.

Lemma 3.3 *Let $\mathfrak{f}$ be a complete Hecke eigenform, $\mathfrak{g}$ and $\mathfrak{h}$ be defined as in (3.3). For any integer $m = tr^2$ where $t \geqslant 1$ is squarefree, we have*

$$\lambda_f(m) \ll_{\mathfrak{f}} |\lambda_f(t)|\tau(r)^2 + |\lambda_{\mathfrak{f}}(t)|\tau(r)^2 \ll_{\mathfrak{f},\varrho} t^{\varrho}\tau(r)^2 \tag{3.18}$$

for $f = \mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ respectively, where ϱ satisfies (3.13). The first implied $\ll$-constant depends only $\mathfrak{f}$ and the second implied $\ll$-constant depends at most on $\mathfrak{f}$ and ϱ.

Remark 4 When $\mathfrak{f}$ lies in the Kohnen plus space, the Hecke operator $\mathsf{T}^+(2^2) := \frac{3}{2}U_4\mathrm{pr}$ is taken in place of $\mathsf{T}(2^2)$, where pr is the orthogonal projection onto the plus space, cf. [8, p. 42-43]. If $\mathfrak{f}$ is an eigenform of $\mathsf{T}^+(2^2)$ and $\mathsf{T}(p^2)$ for all odd primes p, then Lemma 3.3 will still be valid. Firstly Lemma 3.1 and (3.11) hold for $\mathfrak{f}$ and hence (3.12). Next we claim (3.12) holds for $\mathfrak{g}$ and $\mathfrak{h}$. For $\mathfrak{g}$, (3.15) holds if $f \in \mathfrak{S}_{\ell+1/2}$, thus $\mathfrak{g}$ is a complete Hecke eigenform so (3.12) holds. Note $2\mu\mathfrak{h} = \mathfrak{f}|_{U_4}$ once $\mathfrak{f}$ is in the plus space, see [7, Proposition 2]; thus $2\mu\lambda_{\mathfrak{h}}(m) = \lambda_{\mathfrak{f}}(4m)$, the claim follows from (3.12) for $\mathfrak{f}$.

4 A preparation

We start with the method of proof in [3] for the set-up. Meanwhile we amend, for the case $2 \parallel d$, the functional equation to relate $\mathfrak{f}$ with $\mathfrak{g}$ (not $\mathfrak{f}_{\frac{1}{2}}$ in [3]), cf. [3, (4.5)] and our Remark 2 (i). Lastly we indicate the vital components for improvement with a first attempt (see Proposition 1 and Remark 6).

Define $\mathbb{1}_{r^2}(n) = 1$ if $r^2 \mid n$ and 0 otherwise. Replace the divisibility condition with additive characters, we can write

$$\mathbb{1}_{r^2}(n) = \frac{1}{r^2} \sum_{u (\mathrm{mod}\, r^2)} \mathrm{e}\left(\frac{nu}{d^2}\right) = \frac{1}{r^2} \sum_{d | r^2} \sideset{}{^*}\sum_{u (\mathrm{mod}\, d)} \mathrm{e}\left(\frac{nu}{d}\right),$$

where $\sum^*_{u (\mathrm{mod}\, d)}$ runs over $u (\mathrm{mod}\, d)$ coprime to d. Recall $\mu(n)^2 = \sum_{r^2 | n} \mu(r)$. When $\sigma > 1$, one thus has

$$L_{\mathfrak{f}}^{\flat}(s) = \sum_{r=1}^{\infty} \frac{\mu(r)}{r^2} \sum_{d | r^2} \sideset{}{^*}\sum_{u (\mathrm{mod}\, d)} L_{\mathfrak{f}}(s, u/d) \tag{4.1}$$

where

$$L_{\mathfrak{f}}(s, u/d) = \sum_{m\geqslant 1} \frac{\lambda_{\mathfrak{f}}(m)\,\mathrm{e}(mu/d)}{m^s}. \tag{4.2}$$

Let us also denote

$$D_r(s) := r^{-2} \sum_{d\mid r^2} \sum_{u\,(\mathrm{mod}\,d)}^{*} L_{\mathfrak{f}}(s, u/d). \tag{4.3}$$

Now each summand $L_{\mathfrak{f}}(s, u/d)$ extends to an entire function (explained below), so the task is to establish the (uniform) convergence of the series in r. Hence this leads to the estimation of $L_{\mathfrak{f}}(s, u/d)$ in terms of r. The method of Hulse et al. is to derive the functional equation of $L_{\mathfrak{f}}(s, u/d)$ and then apply the convexity principle to $D_r(s)$. They gave an estimate for $D_r(s)$ on the line $\sigma = -\varepsilon$ by bounding $L_{\mathfrak{f}}(s, u/d)$ individually. Consequently they proved that

$$D_r(s) \ll r^{2-4\sigma+\varepsilon}(1+|\tau|)^{1-\sigma+2\varepsilon} \quad (-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon). \tag{4.4}$$

To obtain the functional equation of $L_{\mathfrak{f}}(s, u/d)$, one considers for rational q,

$$\Lambda(\mathfrak{f}, q, s) := \int_0^\infty \mathfrak{f}(\mathrm{i}y+q)y^{s+\frac{\ell}{2}-\frac14}\,\frac{\mathrm{d}y}{y} = \frac{\Gamma(s+\frac{\ell}{2}-\frac14)}{(2\pi)^{s+\frac{\ell}{2}-\frac14}} \sum_{m\geqslant 1} \frac{\lambda_{\mathfrak{f}}(m)\,\mathrm{e}(mq)}{m^s}.$$

The integral is absolutely convergent for every $s \in \mathbb{C}$. We define $\Lambda(\mathfrak{g}, q, s)$ and $\Lambda(\mathfrak{h}, q, s)$ in the same way.

Let $q = u/d$ where $(u, d) = 1$ and $d \geqslant 1$. By [3, Lemma 4.3], $\Lambda(\mathfrak{f}, u/d, s)$ satisfies a functional equation in connection with $\Lambda(\mathfrak{f}, -\overline{u}/d, 1-s)$ and $\Lambda(\mathfrak{h}, -\overline{4u}/d, 1-s)$ respectively according as $4 \mid d$ or $2 \nmid d$, where $x\overline{x} \equiv 1 \ (\mathrm{mod}\ d)$. For the case $2 \parallel d$, we revise $\mathfrak{f}_{\frac12}$ to be $\mathfrak{g}$, which causes a minor change of $\Lambda(\mathfrak{f}_{\frac12}, -\overline{u}/d, 1-s)$ into $\Lambda(\mathfrak{g}, -\overline{u}/(4d), 1-s)$. We would unite the three functional equations into one. Let us introduce

$$q_d = d \ \text{ or } \ 2d \ \text{ according to } 4 \mid d \text{ or not}, \tag{4.5}$$

and the symbols $\lambda(n; d)$ and $\varpi_d(n, v)$ defined as:

	$\lambda(n; d)$	$\varpi_d(n, v)$
$4 \mid d$	$\lambda_{\mathfrak{f}}(n)$	$\varepsilon_v^{2\ell+1}\left(\frac{d}{v}\right)\mathrm{e}\left(\frac{-nv}{d}\right)$
$2 \parallel d$	$\lambda_{\mathfrak{g}}(n)$	$\varepsilon_v^{2\ell+1}\left(\frac{d}{v}\right)\mathrm{e}\left(\frac{-nv}{4d}\right)$
$2 \nmid d$	$\lambda_{\mathfrak{h}}(n)$	$\mathrm{i}^{\ell+1/2}\varepsilon_d^{-(2\ell+1)}\left(\frac{v}{d}\right)\mathrm{e}\left(\frac{-\overline{4}nv}{d}\right)$

(4.6)

with $4\overline{4} \equiv 1 \ (\mathrm{mod}\ d)$. Write

$$L_\infty(s) := (2\pi)^{-s}\Gamma\big(s+\tfrac{\ell}{2}-\tfrac14\big) \tag{4.7}$$

and

$$\widetilde{L}_{\mathfrak{f}}(s, v/d) := \sum_{n\geqslant 1} \lambda(n; d)\varpi_d(n, v)n^{-s}. \tag{4.8}$$

Now we rephrase [3, Lemma 4.3] of Hulse et al. with the above modification for $2 \parallel d$.

Lemma 4.1 *Let $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ where $\ell \geqslant 1$ be an integer, $d \in \mathbb{N}$ and $(u, d) = 1$. Then $L_{\mathfrak{f}}(s, u/d)$ extends analytically to an entire function and satisfies the functional equation:*

$$q_d^s L_\infty(s) L_{\mathfrak{f}}(s, u/d) = \mathrm{i}^{-(\ell+1/2)} q_d^{1-s} L_\infty(1-s)\widetilde{L}_{\mathfrak{f}}(1-s, v/d) \tag{4.9}$$

where $uv \equiv 1 \pmod{d}$.

Remark 5 For the case $2\|d$, the right-side of the equation (4.9) is of period d or probably its divisor in the parameter v, which is not obvious in view of the factor $\mathrm{e}(-\frac{nv}{4d})$. Indeed, one checks that $\varpi_d(n, v+d) = \varpi_d(n, v)$ by using (i) if $d = 2h$ where h is odd, then $\left(\frac{d}{v}\right) = \left(\frac{-2}{h}\right)\left(\frac{h}{v}\right)$; (ii) $n \equiv (-1)^\ell \pmod{4}$ in light of the support of $\{\lambda_{\mathfrak{g}}(n)\}$.

Assume $\sigma < 0$. Applying the functional equation (4.9) to (4.3), we obtain

$$D_r(s) = \mathrm{i}^{-(\ell+1/2)} r^{-2} \frac{L_\infty(1-s)}{L_\infty(s)} \sum_{d|r^2} q_d^{1-2s} \sum_{n\geqslant 1} \frac{\lambda(n;d)}{n^{1-s}} \sideset{}{^*}\sum_{u \,(\mathrm{mod}\, d)} \varpi_d(n,v). \tag{4.10}$$

With a change of running index into v (as $uv \equiv 1 \pmod{d}$), we observe from (4.6) that the sum over u mod d is a particular case of Kloosterman-Salié sums, see [4, Sect. 3]. Immediately we have the Weil bound,

$$\sideset{}{^*}\sum_{u \,(\mathrm{mod}\, d)} \varpi_d(n,v) \ll d^{1/2}\tau(d)(d,n)^{1/2}. \tag{4.11}$$

But in fact it carries more arithmetic properties, as shown below.

Lemma 4.2 *For* $e \in \{0,1,2\}$, *b an odd squarefree integer and* $(a, 2b) = 1$, *we have*

$$\sideset{}{^*}\sum_{v (\mathrm{mod}\, 2^e a^2 b)} \varpi_{2^e a^2 b}(m, v) = G_{e,b}(m) a^2 \sum_{f|a^2} \frac{\mu(f)}{f} \mathbb{1}_{a^2/f}(m), \tag{4.12}$$

where $G_{e,b}(m) \ll \sqrt{b}$ *with an absolute* $\ll$*-constant, and* $\mathbb{1}_d(n) = 1$ *if* $d \mid n$ *and* 0 *otherwise. (Recall that we are confined to* $m \equiv (-1)^\ell \pmod{4}$ *in the case of* $e = 1$.*)*

Lemma 4.2's proof is postponed to Sect. 6. Now we apply (4.11) to give a technically lightweight improvement on the result (4.4) of Hulse et al.

Let $\varepsilon > 0$ be small and $\sigma = -\varepsilon$. Applying (4.11) and Stirling's formula to (4.10), it follows that (recalling $q_d = d$ or $2d$)

$$\begin{aligned} D_r(-\varepsilon + \mathrm{i}\tau) &\ll r^{-2}(1+|t|)^{1+\varepsilon} \sum_{d|r^2} d^{3/2+3\varepsilon} \sum_{n\geqslant 1} |\lambda(n;d)|(d,n)^{1/2} n^{-(1+\varepsilon)} \\ &\ll (r(1+|t|))^{1+\varepsilon} \end{aligned}$$

because $|\lambda(n;d)|(n,d)^{1/2} \leqslant |\lambda(n;d)|^2 + (n,d)$, implying that the last summation is

$$\ll \sum_{n\geqslant 1} |\lambda(n;d)|^2 n^{-(1+\varepsilon)} + \sum_{\ell|d} \sum_{n\geqslant 1} n^{-(1+\varepsilon)} \ll d^\varepsilon.$$

By [3, Lemma 4.2], we have $D_r(1+\varepsilon+\mathrm{i}\tau) \ll r^{-2}$. An application of Phragmén–Lindelöf principle gives

$$D_r(\sigma + \mathrm{i}\tau) \ll r^{1-3\sigma+\varepsilon}(1+|\tau|)^\varepsilon.$$

To assure the convergence in (4.1), we require $1-3\sigma < -1$ and hence conclude the following.

Proposition 1 $L_{\mathfrak{f}}^\flat(\sigma + \mathrm{i}\tau) \ll_{\mathfrak{f},\varepsilon} (|\tau|+1)^{1-\sigma+2\varepsilon}$ *for* $\frac{2}{3} + \varepsilon \leqslant \sigma \leqslant 1+\varepsilon$ *and* $\tau \in \mathbb{R}$.

Remark 6 We have applied only the mean square estimate (3.6) for $\mathfrak{g}$ and $\mathfrak{h}$, and only the Hecke eigenform property of $\mathfrak{f}$ is used. In the next section, we will invoke the arithmetic property revealed in (4.12), the eigenform properties of all $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ and the approximate functional equation to prove the main result.

5 Proof of Theorem 1

We begin with the approximate functional equation for $L_{\mathfrak{f}}(s, u/d)$ below, whose proof is given in Sect. 7.

Lemma 5.1 *Let $T \geqslant 1$ be any number and $s = \sigma + \mathrm{i}\tau$. Suppose $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{2}$ and $|\tau| \leqslant T$. We have*

$$L_{\mathfrak{f}}(s, u/d) = \sum_{m\geqslant 1} \frac{\lambda_{\mathfrak{f}}(m)\,\mathrm{e}(mu/d)}{m^s} V\Big(\frac{m}{q_d T}\Big) + \mathrm{i}^{-(\ell+1/2)}(q_d T)^{1-2s} \sum_{m\geqslant 1} \frac{\lambda(m;d)\varpi_d(m,v)}{m^{1-s}} V_{s,T}\Big(\frac{m}{q_d T}\Big)$$

where $uv \equiv 1 \,(\mathrm{mod}\, d)$, $V(y)$ and $V_{s,T}(y)$ are smooth functions on $(0,\infty)$ and satisfy the following: for any $0 < \eta < \frac{1}{4}$,

$$V(y) = 1 + O_\eta(y^\eta),$$
$$V_{s,T}(y) = \frac{L_\infty(1-s)}{T^{1-2s}L_\infty(s)} + O_\eta(y^\eta) \ll_\eta 1 + y^\eta,$$

and for any $\eta > 0$, both $V(y)$ and $V_{s,T}(y) \ll_\eta y^{-\eta}$.

Now we deal with $L_{\mathfrak{f}}^{\flat}(s)$. In (4.1), we replace the even squarefree r by $2r$ and thus

$$L_{\mathfrak{f}}^{\flat}(s) = \sum_{\substack{r\geqslant 1\\ \text{odd}}} \mu(r) D_r(s) - \sum_{\substack{r\geqslant 1\\ \text{odd}}} \mu(r) D_{2r}(s).$$

Next we separate the sum over d according as $4 \mid d$, $2 \,\|\, d$ or $2 \nmid d$, and hence obtain a decomposition of $L_{\mathfrak{f}}^{\flat}(s)$ into three pieces,

$$L_{\mathfrak{f}}^{\flat}(s) = M_\infty(\mathfrak{f}, s) + M_{1/2}(\mathfrak{f}, s) + M_0(\mathfrak{f}, s),$$

where

$$M_\infty(\mathfrak{f}, s) := -\frac{1}{4} \sum_{\substack{r=1\\ \text{odd}}}^{\infty} \frac{\mu(r)}{r^2} \sum_{d \mid r^2} \sideset{}{^*}\sum_{u (\mathrm{mod}\, 4d)} L_{\mathfrak{f}}(s, u/4d),$$

$$M_{1/2}(\mathfrak{f}, s) := -\frac{1}{4} \sum_{\substack{r=1\\ \text{odd}}}^{\infty} \frac{\mu(r)}{r^2} \sum_{d \mid r^2} \sideset{}{^*}\sum_{u (\mathrm{mod}\, 2d)} L_{\mathfrak{f}}(s, u/2d),$$

$$M_0(\mathfrak{f}, s) := \frac{3}{4} \sum_{\substack{r=1\\ \text{odd}}}^{\infty} \frac{\mu(r)}{r^2} \sum_{d \mid r^2} \sideset{}{^*}\sum_{u (\mathrm{mod}\, d)} L_{\mathfrak{f}}(s, u/d).$$

We shall verify the uniform convergence for the three series of holomorphic functions in $\Re e\, s > \frac{1}{2}$, and concurrently obtain the desired upper estimate (2.3).

Let $\sigma_0 = \frac{1}{2} + \varepsilon_0$ where $\varepsilon_0 > 0$ is arbitrarily small but fixed, and $T \geqslant 1$ be any integer. Consider $s = \sigma + \mathrm{i}\tau$ where $\sigma_0 \leqslant \sigma \leqslant \sigma_0 + \frac{1}{2}$ and $T - 1 \leqslant |\tau| \leqslant T$. In view of the condition

$d \mid r^2$ for squarefree r, we decompose into $d = a^2b$ and $r = abc$ where a, b, c are pairwise coprime and squarefree. It is equivalent to consider the series

$$\sum_{a,b,c} \frac{\mu(2abc)}{(abc)^2} \sideset{}{^*}\sum_{u(\bmod 2^e a^2 b)} L_{\mathfrak{f}}\Big(s, \frac{u}{2^e a^2 b}\Big)$$

where $e = 0, 1, 2$. Now we apply Lemma 5.1 and observe, as before, the set of v given by $uv \equiv 1 \pmod{d}$ runs through a reduced residue class as u varies. We are led to

$$\Sigma_1 = \sum_{a,b,c} \frac{\mu(2abc)}{(abc)^2} \sum_{m\geqslant 1} \frac{\lambda_{\mathfrak{f}}(m)}{m^s} V\left(\frac{m}{a^2bT_e}\right) \sideset{}{^*}\sum_{u \,(\bmod 2^e a^2 b)} \mathrm{e}\left(\frac{mu}{2^e a^2 b}\right), \tag{5.1}$$

$$\Sigma_2 = T_e^{1-2s} \sum_{a,b,c} \frac{\mu(2abc)}{a^{4s} b^{1+2s} c^2} \sum_{m\geqslant 1} \frac{\lambda_{\mathfrak{f},e}(m)}{m^{1-s}} V_{s,T}\left(\frac{m}{a^2bT_e}\right) \sideset{}{^*}\sum_{v(\bmod 2^e a^2 b)} \varpi_{2^e a^2 b}(m, v), \tag{5.2}$$

where $\lambda_{\mathfrak{f},e}(m) := \lambda(m; 2^e)$, see (4.6), and $T_e = 2T$ or $4T$ according as $e = 0$ or not (so that $q_{2^e a^2 b} T = a^2 b T_e$, see (4.5)).

Inserting (4.12) into Σ_2 in (5.2), we further decompose $a = fg$ and $m = fg^2h$ in light of the squarefreeness of f and the conditions $f \mid a^2$ and $(a^2/f) \mid m$.

$$\Sigma_2 = T_e^{1-2s} \sum_{f,g,b,c} \frac{\mu(f)\mu(2fgbc)}{f^{3s} g^{2s} b^{1+2s} c^2} \sum_{h\geqslant 1} \frac{\lambda_{\mathfrak{f},e}(fhg^2)}{h^{1-s}} V_{s,T}\left(\frac{h}{fbT_e}\right) G_{e,b}(fhg^2). \tag{5.3}$$

To justify the uniform convergence, it suffices to consider the sum over dyadic ranges: $(f, g, b, c) \sim (F, G, B, C)$, meaning $F \leqslant f < 2F$, etc. Denote by $\Sigma_2^{F,G,B,C}$ the expression on the right-side of (5.3) under this range restriction. We estimate each summand trivially with the bound $G_{e,b}(m) \ll \sqrt{b}$ in Lemma 4.2. A little simplification leads to

$$\Sigma_2^{F,G,B,C} \ll T_e^{1-2\sigma} \sum_{\substack{(f,g,b,c)\\ \sim(F,G,B,C)}} \frac{|\mu(2fgbc)|}{f^{3\sigma} g^{2\sigma} b^{1/2+2\sigma} c^2} \sum_{h\geqslant 1} \frac{|\lambda_{\mathfrak{f},e}(fhg^2)|}{h^{1-\sigma}} \left| V_{s,T}\left(\frac{h}{fbT_e}\right)\right|. \tag{5.4}$$

Next we treat the sum over h in order for the following estimate:[6]

$$\sum_{h\geqslant 1} \frac{|\lambda_{\mathfrak{f},e}(fhg^2)|}{h^{1-\sigma}} \left| V_{s,T}\left(\frac{h}{fbT_e}\right)\right| \ll G^{\varepsilon} (TFB)^{\sigma-1/2+\varepsilon} \sum_{h\leqslant (FBT)^{1+\varepsilon}} \frac{|\lambda_{\mathfrak{f},e}(fh)|}{\sqrt{h}}. \tag{5.5}$$

To establish (5.5), we invoke Lemmas 3.1 and 3.2, to remove g inside $\lambda_{\mathfrak{f},e}(fhg^2)$, and the estimate for $V_{s,T}$. Set $\mathcal{Q}$ to be the set of all primes *not* dividing g, and write $h = qr^2$ where $p^2|q$ implies $p \in \mathcal{Q}$ and $p|r$ implies $p \notin \mathcal{Q}$ (see Remark 3). As $(2f, g) = 1$, $\mathcal{Q}$ contains 2 and all the prime factors of f. Thus $p^2|fq$ implies $p \in \mathcal{Q}$. Thus, $|\lambda_{\mathfrak{f},e}(fhg^2)| = |\lambda_{\mathfrak{f},e}(fq(gr)^2)| \ll_\varepsilon |\lambda_{\mathfrak{f},e}(fq)|(gr)^{\varepsilon}$. From Lemma 5.1, we deduce the estimate

$$V_{s,T}\left(\frac{h}{fbT_e}\right) \ll_\varepsilon \begin{cases} (FBT)^{\varepsilon} & \text{for } h \leqslant (FBT)^{1+\varepsilon}, \\ h^{-2} & \text{otherwise.} \end{cases}$$

[6] Throughout the proof, ε denotes an arbitrarily small positive number whose value may differ, up to our disposal, at each occurrence.

The sum over $h \geqslant (FBT)^{1+\varepsilon}$ is negligible, in fact $\ll (TFGB)^{\varepsilon}$ (for which we may use the crude bound $|\lambda_{\mathfrak{f},e}(fq)| \ll (fq)^{1/2}$ by (3.6)). Consequently, the left side of (5.5) is

$$\ll (FBT)^{\varepsilon} \sum_{qr^2 \leqslant (FBT)^{1+\varepsilon}} g^{\varepsilon} r^{2(\sigma-1)+\varepsilon} |\lambda_{\mathfrak{f},e}(fq)| q^{-(1-\sigma)}$$
$$\ll G^{\varepsilon}(FBT)^{\sigma-1/2+\varepsilon} \sum_{q \leqslant (FBT)^{1+\varepsilon}} |\lambda_{\mathfrak{f},e}(fq)| q^{-1/2}$$

(recalling $\sigma \geqslant \sigma_0 > 1/2$) which is (5.5) after renaming q into h.

Inserting (5.5) into (5.4), we deduce that

$$\Sigma_2^{F,G,B,C} \ll (TFGB)^{\varepsilon} T^{-\sigma+1/2} F^{-2\sigma} G^{-2\sigma+1} B^{-\sigma} C^{-1} \sum_{f \sim F} \sum_{h \leqslant (FBT)^{1+\varepsilon}} |\lambda_{\mathfrak{f},e}(fh)| (fh)^{-1/2}.$$

Write $m = fh$ and note the divisor function $\tau(m) \ll_{\varepsilon} m^{\varepsilon}$. The double sum is

$$\ll_{\varepsilon} (F^2BT)^{\varepsilon} \sum_{m \ll (F^2BT)^{1+\varepsilon}} |\lambda_{\mathfrak{f},e}(m)| m^{-1/2} \ll_{\varepsilon} (F^2BT)^{1/2+\varepsilon},$$

by (3.6). In summary, we get

$$\Sigma_2^{F,G,B,C} \ll_{\varepsilon} (TFGB)^{\varepsilon} T^{1-\sigma} F^{1-2\sigma} G^{1-2\sigma} B^{-\sigma+1/2} C^{-1}.$$

Recall $\sigma_0 = \frac{1}{2} + \varepsilon_0$ and take $\varepsilon \leqslant \varepsilon_0/2$. Consequently, uniformly for $\sigma_0 \leqslant \sigma \leqslant \sigma_0 + \frac{1}{2}$ and $T - 1 \leqslant |\tau| \leqslant T$, we have $\Sigma_2^{F,G,B,C} \to 0$ as $\max(F, G, B, C) \to \infty$, concluding the uniform convergence. Moreover, as $T^{1-\sigma} \ll (1 + |s|)^{1-\sigma}$, it follows that

$$\Sigma_2 \ll \sum_{F,G,B,C} \Sigma_2^{F,G,B,C} \ll_{\varepsilon} (1 + |s|)^{1-\sigma+\varepsilon_0},$$

recalling the multiple summations range over powers of two.

We turn to Σ_1 in (5.1) which is plainly treated in the same fashion and indeed easier. The inner exponential sum in (5.1) equals

$$2^e a^2 b \sum_{\delta | 2^e a^2 b} \frac{\mu(\delta)}{\delta} \mathbb{1}_{2^e a^2 b/\delta}(m).$$

Noting that a, b are squarefree and $(a, b) = 1$, we write $\delta = 2^j fk$, $a = fg$ and $b = kl$ where $j = 0$ or 1. Then the summation over m will be confined to run over the sequence of $m = 2^{e-j} fg^2 lh$ for positive integers h. Explicitly we have

$$\Sigma_1 = \sum_{j=0,1} \sum_{f,g,k,l,c} \frac{2^{(e-j)(1-s)} \mu(2^j fk) \mu(2fgklc)}{f^{1+s} g^{2s} k^2 l^{1+s} c^2} \sum_{h \geqslant 1} \frac{\lambda_{\mathfrak{f}}(2^{e-j} flhg^2)}{h^s} V\left(\frac{h}{2^j fkT}\right). \tag{5.6}$$

Analogously we divide the summation ranges into dyadic intervals and consider the sub-sum of $\sum_{f,g,k,l,c}$ with $(f, g, k, l, c) \sim (F, G, K, L, C)$. Repeating the above argument,[7] correspondingly we obtain

$$\Sigma_1^{F,G,K,L,C}$$
$$\ll_{\varepsilon} (TFGK)^{\varepsilon} F^{-\sigma-1} G^{1-2\sigma} K^{-1} L^{-\sigma-1} C^{-1} \max_{j=0,1,2} \sum_{(f,l) \sim (F,L)} \sum_{h \ll (FKT)^{1+\varepsilon}} |\lambda_{\mathfrak{f}}(2^j flh)| h^{-\sigma}$$

[7] For the calculation as in (5.5), there is a little variant since the exponent σ of $|h^s|$ is $> \frac{1}{2}$.

$$\ll_\varepsilon (TFGKL)^\varepsilon F^{-1}G^{1-2\sigma}K^{-1}L^{-1}C^{-1} \sum_{m\ll(F^2KLT)^{1+\varepsilon}} |\lambda_{\mathfrak{f}}(m)|m^{-\sigma}$$
$$\ll_\varepsilon (TFGKL)^\varepsilon T^{1-\sigma}F^{1-2\sigma}G^{1-2\sigma}K^{-\sigma}L^{-\sigma}C^{-1},$$

which assures the uniform convergence and the upper estimate. Our proof is complete by changing ε_0 into 2ε.

6 Proof of Lemma 4.2

First consider $e = 1$ or 2, and take the complex conjugate of the left side to simplify a bit the exponential factor. Then by (4.6),

$$\sideset{}{^*}\sum_{v\,(\mathrm{mod}\,2^e a^2 b)} \overline{\varpi_{2^e a^2 b}(m, v)} = \sideset{}{^*}\sum_{v\,(\mathrm{mod}\,2^e a^2 b)} \varepsilon_v^{-(2\ell+1)} \left(\frac{2^e a^2 b}{v}\right) \mathrm{e}\left(\frac{mv}{2^{4-e}a^2 b}\right). \tag{6.1}$$

We write $v = \alpha 8b + \beta a^2$. Note that v runs over a reduced residue class mod $2^e a^2 b$ when α (mod a^2) and β (mod $2^e b$) run over the respective reduced residue classes, since a is odd and $(a, 2b) = 1$. Our substitution choice implies $v \equiv \beta$ (mod 4) and thus $\varepsilon_v = \varepsilon_\beta$. Moreover, the extended Jacobi symbol may be written as, cf. [14, p. 442 (ii)–(iv)],

$$\left(\frac{2^e a^2 b}{v}\right) = \left(\frac{2^e b}{\beta}\right) = \left(\frac{(-1)^{(b-1)/2} 2^e}{\beta}\right)\left(\frac{(-1)^{(b-1)/2} b}{\beta}\right) = \psi_{2,b}(\beta)\chi_{b'}(\beta), \quad \text{(say)},$$

where $b' = (-1)^{(b-1)/2} b$ (is a quadratic discriminant) and $\chi_{b'}(\cdot)$ is the primitive quadratic character of conductor b. (Note b is odd squarefree.) Thus, we express the right side of (6.1) as

$$G'_{e,b}(m) \sideset{}{^*}\sum_{\alpha\ (\mathrm{mod}\,a^2)} \mathrm{e}\left(\frac{2^{e-1} m\alpha}{a^2}\right) = G'_{e,b}(m) a^2 \sum_{f|a^2} \frac{\mu(f)}{f} \mathbb{1}_{a^2/f}(m)$$

(cf. [6, p.44 (3.2)] and recalling a is odd) where

$$G'_{e,b}(m) = \sideset{}{^*}\sum_{\beta\ (\mathrm{mod}\,2^e b)} \varepsilon_\beta^{-(2\ell+1)} \psi_{2,b}(\beta)\chi_{b'}(\beta)\, \mathrm{e}\left(\frac{m\beta}{2^{4-e}b}\right). \tag{6.2}$$

This gives (4.12) with $G_{e,b}(m) = \overline{G'_{e,b}(m)}$, and thus it remains to show $G'_{e,b}(m) \ll \sqrt{b}$ so as to finish the proof.

We separate the sum in (6.2) into two subsums, whose summands take the same value of ε_β, as follows:

$$\sideset{}{^*}\sum_{\substack{\beta\ (\mathrm{mod}\,2^e b)\\ \beta\equiv 1\,(\mathrm{mod}\,4)}} + \mathrm{i}^{-(2\ell+1)} \sideset{}{^*}\sum_{\substack{\beta\ (\mathrm{mod}\,2^e b)\\ \beta\equiv -1\,(\mathrm{mod}\,4)}} .$$

With the primitive character χ_4 mod 4 (given by $\chi_4(n) = (-1)^{(n-1)/2}$ for odd n), we relax the extra conditions with the factors $\frac{1}{2}(1 + \chi_4(\beta))$ and $\frac{1}{2}(1 - \chi_4(\beta))$. Consequently, letting $\theta_\pm = \frac{1}{2}(\mathrm{i}^{\ell+1/2} \pm \mathrm{i}^{-(\ell+1/2)})$, we rearrange the terms to have

$$G'_{e,b}(m) = \theta_+ \sideset{}{^*}\sum_{\beta\ (\mathrm{mod}\,2^e b)} \psi_{2,b}(\beta)\chi_{b'}(\beta)\, \mathrm{e}\left(\frac{m\beta}{2^{4-e}b}\right) + \theta_- \sideset{}{^*}\sum_{\beta\ (\mathrm{mod}\,2^e b)} \psi'_{2,b}(\beta)\chi_{b'}(\beta)\, \mathrm{e}\left(\frac{m\beta}{2^{4-e}b}\right)$$

where $\psi'_{2,b}(\beta) = \chi_4\psi_{2,b}$. Both $\psi_{2,b}$ and $\psi'_{2,b}$ are characters (not necessarily primitive) modulo 8. Repeating the argument of writing $\beta = 8\beta_1 + b\beta_2$, we infer that

$$G'_{e,b}(m) \ll \left| \sideset{}{^*}\sum_{\beta \,(\mathrm{mod}\, b)} \chi_{b'}(\beta)\, \mathrm{e}\left(\frac{2^{e-1}m\beta}{b}\right)\right| \ll b^{1/2} \tag{6.3}$$

by the primitivity of $\chi_{b'}$, see [6, p. 47 (3.12) and p. 48 (3.14)].

Next we come to the case $e = 0$. In this case, we set $v = b\alpha + a^2\beta$ with $\alpha \,(\mathrm{mod}\, a^2)$, $(\alpha, a) = 1$ and $\beta \,(\mathrm{mod}\, b)$, $(\beta, b) = 1$, then

$$\sideset{}{^*}\sum_{v (\mathrm{mod}\, a^2 b)} \left(\frac{v}{b}\right) \mathrm{e}\left(\frac{-\overline{4}mv}{a^2 b}\right) = a^2 \sum_{f | a^2} \frac{\mu(f)}{f} \mathbb{1}_{a^2/f}(m) \sideset{}{^*}\sum_{\beta (\mathrm{mod}\, b)} \chi_{b'}(\beta)\, \mathrm{e}\left(-\frac{\overline{4}m\beta}{b}\right).$$

Take $G_{0,b}(m)$ to be the product of $\mathrm{i}^{\ell+1/2}\varepsilon_d^{-(2\ell+1)}$ and the character sum (over β). This gives, with (6.3), the desired result in (4.12), completing the proof.

7 Proof of Lemma 5.1

Let $H(z)$ be an entire function such that $H(z) \ll_{\eta,A} (1+|z|)^{-A}$ for $\Re e\, z = \eta$ and any $A > 0$, $H(0) = 1$ and $H(z) = H(-z)$. (See [2] for its construction.) We infer with the residue theorem that

$$L_{\mathfrak{f}}(s, u/d) = \frac{1}{2\pi \mathrm{i}} \left\{ \int_{(2)} - \int_{(-2)} \right\} L_{\mathfrak{f}}(s+z, u/d)(q_d T)^z H(z) \frac{\mathrm{d}z}{z} =: I_1 + I_2, \qquad \text{(say)}.$$

Changing z to $-z$ and invoking the functional equation, we transform I_2 into

$$\mathrm{i}^{-(\ell+1/2)}(q_d T)^{1-2s} \frac{1}{2\pi \mathrm{i}} \int_{(2)} \widetilde{L}_{\mathfrak{f}}(1-s+z, v/d) \frac{L_\infty(1-s+z)}{T^{1-2s+2z}L_\infty(s-z)} (q_d T)^z H(z) \frac{\mathrm{d}z}{z}.$$

Set

$$V(y) := \frac{1}{2\pi \mathrm{i}} \int_{(2)} y^{-z} H(z) \frac{\mathrm{d}z}{z},$$
$$V_{s,T}(y) := \frac{1}{2\pi \mathrm{i}} \int_{(2)} \frac{L_\infty(1-s+z)}{T^{1-2s+2z}L_\infty(s-z)} y^{-z} H(z) \frac{\mathrm{d}z}{z}.$$

The formula follows readily after inserting the Dirichlet series of $L_{\mathfrak{f}}(s, u/d)$ and $\widetilde{L}_{\mathfrak{f}}(s, v/d)$ in (4.2) and (4.8). It remains to check the properties of $V(y)$ and $V_{s,T}(y)$. The case of $V(y)$ is quite obvious, and for $V_{s,T}(y)$, we recall the estimate in [2, Lemma 3.2]: For $\alpha > -\sigma$,

$$\frac{\Gamma(z+\sigma)}{\Gamma(z)} \ll_{\alpha,\sigma} |z+\sigma|^\sigma \qquad (\Re e\, z \geqslant \alpha).$$

Recalling (4.7), this yields

$$\frac{L_\infty(1-s+z)}{T^{1-2s+2z}L_\infty(s-z)} \ll_\eta \left| \frac{1-s+z+\frac{\ell}{2}-\frac{1}{4}}{T} \right|^{1-2\sigma+2\eta} \ll \left(1+\frac{|z|}{T}\right)^{1-2\sigma+2\eta}. \tag{7.1}$$

We shift the line of integration to the right, yielding $V_{s,T}(y) \ll_\eta y^{-\eta}$ for any $\eta > 0$ and shift to the left to derive

$$V_{s,T}(y) = \frac{L_\infty(1-s)}{T^{1-2s}L_\infty(s)} + O_\eta(y^\eta)$$

for any $0 < \eta < \frac{1}{4}$. The main term is $O(1)$ by (7.1). The proof of Lemma 5.1 ends.

8 Proof of Theorem 2

We start with a lemma which supersedes [9, (14)] with an improvement on the exponent of the O-term from $\frac{3}{4} + \varrho + \varepsilon$ to $\frac{3}{4} + \varepsilon$.

Lemma 8.1 *Let $\ell \geqslant 2$ be a positive integer and $\mathfrak{f} \in \mathfrak{S}_{\ell+1/2}$ be a complete Hecke eigenform. Then for any $\varepsilon > 0$ and all $x \geqslant 2$, we have*

$$\sum_{n \leqslant x} |\lambda_{\mathfrak{f}}(n)|^2 = D_{\mathfrak{f}}\, x + O_{\mathfrak{f},\varepsilon}\big(x^{3/4+\varepsilon}\big), \tag{8.1}$$

where $D_{\mathfrak{f}}$ is a positive constant depending on $\mathfrak{f}$.

Proof We choose two smooth compactly supported functions $w_\pm$ such that

- $w_-(x) = 1$ for $x \in [X+Y, 2X-Y]$, $w_-(x) = 0$ for $x \geqslant 2X$ and $x \leqslant X$;
- $w_+(x) = 1$ for $x \in [X, 2X]$, $w_+(x) = 0$ for $x \geqslant 2X+Y$ and $x \leqslant X-Y$;
- $w_\pm^{(j)}(x) \ll_j Y^{-j}$ for all $j \geqslant 0$;
- the Mellin transform of $w(x)$ is

$$\begin{aligned}\widehat{w_\pm}(s) &:= \int_0^\infty w_\pm(x) x^{s-1}\,\mathrm{d}x \\ &= \frac{1}{s\cdots(s+j-1)} \int_0^\infty w_\pm^{(j)}(x) x^{s+j-1}\,\mathrm{d}x \\ &\ll_j \frac{Y}{X^{1-\sigma}} \left(\frac{X}{|s|Y}\right)^j \quad \forall j \geqslant 1;\end{aligned} \tag{8.2}$$

- trivially $\widehat{w_\pm}(s) \ll X^\sigma$ and

$$\widehat{w_\pm}(1) = X + O(Y). \tag{8.3}$$

Obviously we have

$$\sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_-(n) \leqslant \sum_{X<n\leqslant 2X} |\lambda_{\mathfrak{f}}(n)|^2 \leqslant \sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_+(n). \tag{8.4}$$

Let the Dirichlet series associated with $|\lambda_{\mathfrak{f}}(n)|^2$ be defined as (see e.g. [9, (11)])

$$D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) = \sum_{n=1}^\infty |\lambda_{\mathfrak{f}}(n)|^2 n^{-s}.$$

By the Mellin inversion formula

$$w_\pm(x) = \frac{1}{2\pi\mathrm{i}} \int_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \widehat{w_\pm}(s) x^{-s}\,\mathrm{d}s,$$

we write

$$\sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_\pm(n) = \frac{1}{2\pi\mathrm{i}} \int_{(2)} \widehat{w_\pm}(s) D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)\,\mathrm{d}s.$$

With the help of Cauchy's residue theorem, we obtain that

$$\sum_n \lambda_{\mathfrak{f}}(n)^2 w_\pm(n) = D_{\mathfrak{f}} \widehat{w_\pm}(1) + \frac{1}{2\pi \mathrm{i}} \int_{(\kappa)} \widehat{w_\pm}(s) D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)\, \mathrm{d}s, \tag{8.5}$$

where $\frac{1}{2} < \kappa < 1$ and $D_{\mathfrak{f}} := \mathrm{Res}_{s=1} D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s)$. By (8.3), (8.2) with $j = 2$ and the convexity bound [9, Proposition 7]

$$D(\mathfrak{f} \otimes \bar{\mathfrak{f}}, s) \ll_{\mathfrak{f},\varepsilon} (1 + |\tau|)^{2\max(1-\sigma,0)+\varepsilon} \qquad (\tfrac{1}{2} < \sigma \leqslant 3),$$

we derive

$$\sum_n |\lambda_{\mathfrak{f}}(n)|^2 w_\pm(n) = D_{\mathfrak{f}} X + O_{\mathfrak{f},\varepsilon}\big(Y + X^{1+\kappa} Y^{-1}\big).$$

Taking $\kappa = \frac{1}{2} + \varepsilon$ and $Y = X^{3/4}$, and combining the obtained estimation with (8.4), we find that

$$\sum_{X < n \leqslant 2X} |\lambda_{\mathfrak{f}}(n)|^2 = D_{\mathfrak{f}} X + O_{\mathfrak{f},\varepsilon}\big(X^{3/4+\varepsilon}\big),$$

which implies (8.1) after a dyadic summation. □

Now we are ready to prove Theorem 2 along the same line of argument in [9]. Take $h = x^\eta$ where $\eta > \frac{3}{4}$ is specified later. Lemma 8.1 gives

$$\text{(i)} \quad Ch \leqslant \sum_{x \leqslant n \leqslant x+h} \lambda_{\mathfrak{f}}(n)^2 \quad \text{and} \quad \text{(ii)} \quad \sum_{x/m^2 \leqslant t \leqslant (x+h)/m^2} \lambda_{\mathfrak{f}}(n)^2 \ll hm^{-3/2}$$

for any $m \leqslant \sqrt{x+h}$, where the positive constant C and the implied $\ll$-constant depend on $\mathfrak{f}$ and η only. Combining (i) with the bound $\lambda_{\mathfrak{f}}(tm^2) \ll \lambda_{\mathfrak{f}}(t)\tau(m)^2$ (cf. [9, Lemma 6]) leads to

$$Ch \leqslant \sum_{x \leqslant n \leqslant x+h} \lambda_{\mathfrak{f}}(n)^2 \leqslant C' \sum_{m \leqslant \sqrt{x+h}} \tau(m)^4 \sideset{}{^\flat}\sum_{x/m^2 \leqslant t \leqslant (x+h)/m^2} \lambda_{\mathfrak{f}}(t)^2$$

where $C' > 0$ is a constant depending at most on $\mathfrak{f}$. By (ii) and the fact

$$\sum_{m \geqslant A} \tau(m)^4 m^{-3/2} \gg A^{-1/2+\varepsilon},$$

we conclude that for a large enough constant A,

$$\sum_{m \leqslant A} \tau(m)^4 \sideset{}{^\flat}\sum_{x/m^2 \leqslant t \leqslant (x+h)/m^2} \lambda_{\mathfrak{f}}(t)^2 \geqslant \{C/C' + O(A^{-1/2+\varepsilon})\}h \gg h$$

which is [9, (23)]. Thus, repeating the same argument (in [9, (24)-(26)]), we obtain [9, (26)] with a smaller admissible $h = x^\eta$ (here only $\eta > \frac{3}{4}$ is required instead of $\eta > \frac{3}{4} + \varrho$, which is due to the improved O-term in Lemma 8.1).

Next we apply the new bound (2.3) of Theorem 1 to $M(\mathfrak{f}, s)$ ($= L_{\mathfrak{f}}^\flat(s)$ here, cf. [9, (12)]) in Equation (20) of [9, Sect. 4.2]. Then we get

$$\sideset{}{^\flat}\sum_{x \leqslant t \leqslant x+h} \lambda_{\mathfrak{f}}(t) \min\left\{\log\left(\frac{x+h}{t}\right), \log\left(\frac{t}{x}\right)\right\} \ll_\varepsilon h^{\frac{1}{2}} x^\varepsilon$$

and subsequently improve the upper bound $O(h^{3/4}x^{\varepsilon})$ in [9, (21)] to $O(h^{1/2}x^{\varepsilon})$. Ultimately the lower bound in [9, (27)], which relies on [9, (21) and (26)], is sharpened to

$$x^{-1-\varrho}h^2 + O(h^{1/2}x^{\varepsilon}). \tag{8.6}$$

The optimal choice of η for the positivity of (8.6) is $\frac{2}{3}(1+\varrho)+\varepsilon$. Together with the constraint $\eta > \frac{3}{4}$ (from the new smaller admissible range of h), we set

$$\eta = \max\left\{\tfrac{2}{3}(1+\varrho), \tfrac{3}{4}\right\} + \varepsilon.$$

The proof is complete with the same argument in remaining part of [9, Sect. 4.2].

Acknowledgements Jiang is supported by the China Postdoctoral Science Foundation (No. 2017M620285), the Natural Science Foundation of Shandong Province (No. ZR2018QA004) and NSFC (No. 11801318), and Lü is supported by NSFC (Nos. 11771252, 11531008), IRT16R43, and Taishan Scholars Project. Lü, Royer and Wu are partially supported by the program PRC 1457 - AuForDiP (CNRS-NSFC). Lau is supported by GRF (Project no. 17313616 and 17305617) of the Research Grants Council of Hong Kong. The preliminary form of this paper was finished during the visit of E. Royer and J. Wu at The University of Hong Kong in 2015. They would like to thank the department of mathematics for hospitality and excellent working conditions.

References

1. Conrey, J.B., Iwaniec, H.: The cubic moment of central values of automorphic L-functions. Ann. Math. **151**, 1175–1216 (2000)
2. Harcos, G.: Uniform approximate functional equation for principal L-functions. IMRN **18**, 923–932 (2002). (Erratum, ibid. 2004, 659–660)
3. Hulse, T.A., Kiral, E.M., Kuan, C.I., Lim, L.-M.: The sign of Fourier coefficients of half-integral weight cusp forms. Int. J. Number Theory **8**, 749–762 (2012)
4. Iwaniec, H.: Fourier coefficients of modular forms of half-integral weight. Invent. Math. **87**, 385–401 (1987)
5. Iwaniec, H., Luo, W., Sarnak, P.: Low lying zeros of families of L-functions. Inst. Hautes Études Sci. Publ. Math. **91**, 55–131 (2000)
6. Iwaniec, H., Kowalski, E.: Analytic Number Theory. American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI (2004)
7. Kohnen, W.: Modular forms of half-integral weight on $\Gamma_0(4)$. Math. Ann. **248**, 249–266 (1980)
8. Kohnen, W.: Newforms of half-integral weight. J. Reine Angew. Math. **333**, 32–72 (1982)
9. Lau, Y.-K., Royer, E., Wu, J.: Sign of Fourier coefficients of modular forms of half integral weight. Mathematika **62**, 866–883 (2016)
10. Matomäki, K., Radziwill, M.: Multiplicative functions in short intervals. Ann. Math. **183**, 1015–1056 (2016)
11. Niwa, S.: On Shimura's trace formula. Nagoya Math. J. **66**, 183–202 (1977)
12. Ono, K.: The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series, CBMS Regional Conference Series in Mathematics, vol. 102. American Mathematical Society, Providence, RI (2004)
13. Purkait, S.: On Shimura's decomposition. Int. J. Number Theory **9**, 1431–1445 (2013)
14. Shimura, G.: On modular forms of half-integral weight. Ann. Math. **97**, 440–481 (1973)