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Rankin-Cohen brackets on quasimodular forms

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Abstract. We give the algebra of quasimodular forms a collection of Rankin-Cohen operators. These operators extend those defined by Cohen on modular forms and, as for modular forms, the first of them provides a Lie structure on quasimodular forms. They also satisfy a "Leibniz rule" for the usual derivation. Rankin-Cohen operators are useful for proving arithmetical identities. In particular, we explain why Chazy equation has the exact shape it has.

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Introduction

The purpose of this paper is to present a generalisation for quasimodular forms of the Rankin-Cohen brackets for modular forms: for each $n \geq 0$, k, ℓ, s, t positive integers, we define bilinear differential operators $[\,,\,]_n$ sending $\widetilde{M}_k^{\leq s} \times \widetilde{M}_\ell^{\leq t}$ to $\widetilde{M}_{k+\ell+2n}^{\leq s+t}$. We have denoted $\widetilde{M}_k^{\leq s}$ the vector space of quasimodular forms of weight k and depth less or equal than s on $\mathrm{SL}(2,\mathbb{Z})$ (see section 1.1 for the definitions).

We give a quite precise description of the image of this bilinear form in terms of modular and parabolic forms. This allows us to obtain efficiently classical differential equations and arithmetical identities.

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Then we prove that the Rankin-Cohen brackets satisfy the "Leibniz rule" for the normalized usual derivation $\left(\mathrm{D} := \frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}\right) : \mathrm{D}[f,g]_n = [\mathrm{D}\, f, g]_n + [f, \mathrm{D}\, g]_n$.

The first section is a presentation of the definitions and classical results concerning quasimodular forms and Rankin-Cohen brackets on modular forms.

In the second section, we prove the following theorem.

Theorem 1. *Let k, ℓ in $\mathbb{Z}_{>0}$, $s \in \{0, \dots, \lfloor k/2 \rfloor\}$, $t \in \{0, \dots, \lfloor \ell/2 \rfloor\}$ and $n \in \mathbb{Z}_{\geq 0}$. Define*

$$\Phi_{n;k,s;\ell,t}(f,g) := \sum_{r=0}^{n} (-1)^r \binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r} \mathrm{D}^r f\, \mathrm{D}^{n-r} g.$$

Then

$$\Phi_{n;k,s;\ell,t}(\tilde{M}_k^{\leq s}, \tilde{M}_\ell^{\leq t}) \subset \tilde{M}_{k+\ell+2n}^{\leq s+t}.$$

In some case we get a more precise description in terms of the spaces of modular forms M_k and the spaces of parabolic forms S_k.

Proposition 2. *Under the hypothesis of Theorem 1, if $n > 0$ then*

$$\Phi_{n;k,s;\ell,t}(\tilde{M}_k^{\leq s}, \tilde{M}_\ell^{\leq t}) \in S_{k+\ell+2n} \oplus \bigoplus_{j=1}^{s+t} \mathrm{D}^j M_{k+\ell+2n-2j}.$$

Moreover, if $n > s+t$, then

$$\Phi_{n;k,s;\ell,t}(\tilde{M}_k^{\leq s}, \tilde{M}_\ell^{\leq t}) \in S_{k+\ell+2n} \oplus \bigoplus_{j=1}^{s+t-1} \mathrm{D}^j M_{k+\ell+2n-2j} \oplus \mathrm{D}^{s+t} S_{k+\ell+2n-2s-2t}.$$

The same conclusion holds if $n = s+t$ and $f \in \tilde{M}_k^{\leq s}$ or $g \in \tilde{M}_\ell^{\leq t}$ vanishes at infinity.

Remark 1. This notion is consistent with the one for modular forms, the standard Rankin-Cohen bracket of $f \in M_k$ and $g \in M_\ell$ (see section 1.2 for the definition) is $\Phi_{n;k,0;\ell,0}(f,g)$.

Remark 2. For $n \geq 0$, a bilinear differential operator Ψ sending $\tilde{M}_k^{\leq s} \times \tilde{M}_\ell^{\leq t}$ to $\bigcup_v \tilde{M}_{k+\ell+2n}^{\leq v}$ is necessarily (for weight compatibility reasons) a linear combination of $(f,g) \mapsto \mathrm{D}^r f\, \mathrm{D}^{n-r} g$, $r \in \{0, \dots, n\}$. Such a differential operator sends in principle $\tilde{M}_k^{\leq s} \times \tilde{M}_\ell^{\leq t}$ to $\tilde{M}_{k+\ell+2n}^{\leq s+t+n}$ (see Lemma 7). So the operator Φ introduced in Theorem 1 has the advantage of reducing the depth of the quasimodular form obtained, and it was not obvious that such an operator was existing.

Remark 3. Theorem 1 is valid for quasimodular forms on any subgroup of finite index in $\mathrm{SL}(2, \mathbb{Z})$.

In the third section, we show that the behaviour of this operator under derivation is natural.

Theorem 3. *Under the hypothesis of Theorem 1, for all* $f \in \tilde{M}_k^{\leq s}$ *and* $g \in \tilde{M}_\ell^{\leq t}$,

$$\mathrm{D}\,\Phi_{n;k,s;\ell,t}(f, g) = \Phi_{n;k,s;\ell+2,t+1}(f, \mathrm{D}\,g) + \Phi_{n;k+2,s+1;\ell,t}(\mathrm{D}\,f, g).$$

Remark 4. For f of weight k and *exact depth* s and g of weight ℓ and *exact depth* t, we write $[f, g]_n$ instead of $\Phi_{n;k,s;\ell,t}(f, g)$. Recall (see Proposition 6) that if h has weight $w > 0$ and depth d then $\mathrm{D}\,h$ has weight $w + 2$ and depth $d + 1$. The above theorem may then be rewritten as

$$\mathrm{D}[f, g]_n = [\mathrm{D}\,f, g]_n + [f, \mathrm{D}\,g]_n.$$

For modular forms, Cohen, Manin & Zagier [3] showed that the sum of Rankin-Cohen brackets defines an associative product on the algebra $M = \prod_{k \geq 0} M_k$. In a recent paper, Bieliavski, Tang & Yao [1] showed that this sum is isomorphic to the standard Moyal product. Do the Rankin-Cohen brackets for quasimodular forms introduced here have such a geometric interpretation?

The existence of Rankin-Cohen brackets (thanks to Proposition 2) provides a new tool to obtain arithmetical identities. For example, we recover the Ramanujan differential equations, Chazy differential equation (and explain why such a differential equation has to exist), van der Pol equality and Niebur equality. As usual, define for $h \geq 2$ the Eisenstein series:

$$E_h(z) := 1 - \frac{2h}{B_h} \sum_{n=1}^{+\infty} \sigma_{h-1}(n) \exp(2\pi \mathrm{i} n z) \tag{1}$$

where B_h is the Bernoulli number and

$$\sigma_r(n) := \sum_{d|n} d^r$$

for any positive integer n and any r.

One of the three Ramanujan equations is

$$\mathrm{D}\,E_2 = -\frac{1}{12}(E_4 - E_2^2).$$

It is a direct consequence of

$$[E_2, \Delta]_1 = \Delta E_4$$

where Δ is the unique primitive form of weight 12 on $\mathrm{SL}(2,\mathbb{Z})$. If we write $\tau(n)$ for the nth coefficient of Δ, Niebur [6] equality is

$$\tau(n) = n^4\sigma_1(n) - 24\sum_{a=1}^{n-1}(35a^4 - 52a^3n + 18a^2n^2)\sigma_1(a)\sigma_1(n-a)$$

and it follows from

$$[E_2, E_2]_4 = -48\Delta.$$

Van der Pol [13] equality is

$$\tau(n) = n^2\sigma_3(n) + 60\sum_{a=1}^{n-1} a(9a-5n)\sigma_3(a)\sigma_3(n-a).$$

It follows from

$$[E_4, \mathrm{D}\, E_4]_1 = 960\Delta.$$

Many examples of the two previous type are given in [11]. Finally, a quite astonishing equality is Chazy differential equation. Its usual form is

$$\mathrm{D}^3 E_2 = E_2 \mathrm{D}^2 E_2 - \frac{3}{2}(\mathrm{D}\, E_2)^2$$

and it follows from

$$[[K, \Delta]_1, \Delta]_1 = 24\Delta K^2 \tag{2}$$

where $K = [E_2, \Delta]_1$. The most outer bracket is on modular forms since it may be shown that $[K, \Delta]_1$ has depth 0. That such a differential equation has to exist is a consequence of the following proposition that we prove using Rankin-Cohen brackets.

Proposition 4. *Let $n \geq 0$ and $r \in \{0, \dots, n\}$. Then*

$$\mathrm{D}^r E_2 \mathrm{D}^{n-r} E_2 \in \bigoplus_{\substack{j=0 \\ j\equiv n \pmod 2}}^{n-4} \mathrm{D}^j S_{2n+4-2j} \oplus \mathbb{C}\,\mathrm{D}^n E_4 \oplus \mathbb{C}\,\mathrm{D}^{n+1} E_2.$$

In particular, $[E_2, E_2]_0 \in \mathbb{C}E_4 + \mathbb{C}\,\mathrm{D}\, E_2$, $[E_2, E_2]_2 \in \mathbb{C}\,\mathrm{D}^2 E_4$, $[E_2, E_2]_4 \in \mathbb{C}\Delta$ *and*

$$[E_2, E_2]_{2n} \in S_{4(n+1)} \oplus \mathrm{D}^2 S_{4n} \quad \text{if } n \geq 3.$$

Indeed for $n = 2$, this proposition implies that both quasimodular forms $E_2 \mathrm{D}^2 E_2$ and $(\mathrm{D}\, E_2)^2$ are in $\mathbb{C}\,\mathrm{D}^2 E_4 \oplus \mathbb{C}\,\mathrm{D}^3 E_2$. Hence $\mathrm{Vect}(E_2 \mathrm{D}^2 E_2, (\mathrm{D}\, E_2)^2) = \mathrm{Vect}(\mathrm{D}^2 E_4, \mathrm{D}^3 E_2)$ and $\mathrm{D}^3 E_2$ is a linear combination of $E_2 \mathrm{D}^2 E_2$ and $(\mathrm{D}\, E_2)^2$: this is the shape of Chazy equation.

1. Overview

1.1 Quasimodular forms

In this section, we introduce usual definitions and notations and recall some useful properties of quasimodular forms. For a more detailed introduction, see [5, §17].

We introduce the following notations: as usual, the complex upper half-plane is denoted by $\mathscr{H}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and $z \in \mathscr{H}$, we define

$$\mathrm{X}(\gamma, z) := \frac{c}{cz+d}$$

and

$$\mathrm{X}(\gamma)\colon z \mapsto \mathrm{X}(\gamma, z).$$

For $k \geq 0$, $f\colon \mathscr{H} \to \mathbb{C}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ the function $(f \underset{k}{|} \gamma)$ is defined by $(f \underset{k}{|} \gamma)(z) = (cz+d)^{-k} f(\gamma z)$.

Definition 5. *Let $k \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}_{\geq 0}$. A holomorphic function $f\colon \mathscr{H} \to \mathbb{C}$ is a quasimodular form of weight k, depth s (over $\mathrm{SL}(2,\mathbb{Z})$) if there exist holomorphic functions $Q_0(f)$, $Q_1(f)$, ..., $Q_s(f)$ on $\mathscr{H}$ such that*

$$(f \underset{k}{|} \gamma) = \sum_{i=0}^{s} Q_i(f)\, \mathrm{X}(\gamma)^i \tag{3}$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ and such that $Q_s(f)$ is not identically vanishing and f has no negative terms in its Fourier expansion. By convention, the 0 function is a quasimodular form of depth $-\infty$ and any weight.

Remark 5. Taking $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\gamma = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in (3) implies that f is periodic of period 1 hence has a Fourier expansion. The definition requires this Fourier expansion to be of the shape

$$f(z) = \sum_{n=0}^{+\infty} \hat{f}(n) e^{2\pi i n z}.$$

The set of quasimodular forms of weight k and depth s is denoted by $\tilde{M}_k^s$. It is often more convenient to use the $\mathbb{C}$-vector space of quasimodular forms of weight k and depth less or equal than s, which is denoted by $\tilde{M}_k^{\leq s}$. It can be shown that there are no quasimodular forms (except 0) of negative weight or of depth $s > k/2$ [5, lemme 120]. Hence we extend our notation by defining $\tilde{M}_k^{\leq s} = \{0\}$ if $k < 0$ and $\tilde{M}_k^{\leq s} = \tilde{M}_k^{\leq k/2}$ if $s > k/2$.

Remark 6. With this definition, the space M_k of modular forms of weight k for $\mathrm{SL}(2,\mathbb{Z})$ is exactly the space $\tilde{M}_k^{\leq 0}$.

Remark 7. A basic example of a quasimodular form which is not a modular form is E_2 defined in (1). It satisfies for all $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ the transformation property

$$(E_2 \underset{2}{|} \gamma) = E_2 + \frac{6}{\pi \mathrm{i}} \mathrm{X}(\gamma),$$

proving that $E_2 \in \tilde{M}_2^1$ (see e.g., [5, lemme 19]).

The space $\tilde{M}_* = \bigcup_{k,s} \tilde{M}_k^{\leq s}$ is equipped with a natural filtered-graded algebra structure (the grading according to the weight, the filtration according to the depth). The canonical multiplication $(f,g) \longmapsto fg$ defines a morphism $\tilde{M}_k^{\leq s} \times \tilde{M}_\ell^{\leq t} \longrightarrow \tilde{M}_{k+\ell}^{\leq s+t}$.

If $f \in \tilde{M}_k^{\leq s}$, the sequence $(Q_i(f))_{i\in\mathbb{Z}}$ is defined by the quasimodularity condition (3), if $i \in \{0,\dots,s\}$, and $Q_i(f) = 0$ for $i \notin \{0,\dots,s\}$. One can show that $Q_0(f) = f$ and $Q_i(f) \in \tilde{M}_{k-2i}^{\leq s-i}$ [5, Lemme 119].

Quasimodular forms are the natural extension of modular forms into a stable by derivation space, because of the following proposition.

Proposition 6. *If $k > 0$, the normalized derivation $\mathrm{D} := \frac{1}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z}$ maps $\tilde{M}_k^s$ to $\tilde{M}_{k+2}^{s+1}$.*

For $r \in \mathbb{Z}_{\geq 0}$, write $f^{(r)} := \mathrm{D}^r(f)$ and $f' = f^{(1)}$. The following lemma connects the transformation equation of f and $f^{(r)}$.

Lemma 7. *Let $f \in \tilde{M}_k^{\leq s}$. Then,*

$$(\mathrm{D}^r f \underset{k+2r}{|} \gamma) = \sum_{i=0}^{s+r} \left[\sum_{j=0}^{r} \frac{1}{(2\pi\mathrm{i})^j} j! \binom{r}{j} \binom{k+r-i+j-1}{j} \mathrm{D}^{r-j} Q_{i-j}(f) \right] \mathrm{X}(\gamma)^i \tag{4}$$

for all $r \in \mathbb{Z}_{\geq 0}$ and $\gamma \in \Gamma$.

Proof. The result is obtained inductively on r: it is obvious for $r = 0$, and for the induction suppose that for $r \geq 0$, formula (4) holds. Let $g = f^{(r)}$. For $i \in \mathbb{Z}$ we have

$$Q_i(g) = \sum_{j=0}^{r} \frac{1}{(2\pi\mathrm{i})^j} j! \binom{r}{j} \binom{k+r-i+j-1}{j} Q_{i-j}(f)^{(r-j)} \in \tilde{M}_{k+2r-2i}^{\leq s+r-i}. \tag{5}$$

Then using Proposition 6 (which implies that $f^{(r+1)} \in \tilde{M}_{k+2r+2}^{\leq r+s+1}$) and lemma 118 of [5] we find

$$(f^{(r+1)} \underset{k+2r+2}{|} \gamma) = \sum_{i=0}^{s+r+1} \left(Q_i(g)' + \frac{k+2r-i+1}{2\pi\mathrm{i}} Q_{i-1}(g) \right) \mathrm{X}(\gamma)^i.$$

From (5) we compute

$$\begin{aligned} & Q_i(g)' + \frac{k+2r-i+1}{2\pi\mathrm{i}} Q_{i-1}(g) \\ & \quad = Q_i(f)^{(r+1)} + \frac{k+2r-i+1}{(2\pi\mathrm{i})^{r+1}} r! \binom{k+2r-i}{r} Q_{i-r-1}(f) \\ & \qquad + \sum_{j=1}^{r} \frac{1}{(2\pi\mathrm{i})^j} Q_{i-j}(f)^{(r+1-j)} \\ & \qquad \times \left(\frac{r!}{(r-j)!} \binom{k+r-i+j-1}{j} \right. \\ & \qquad \left. + \frac{(k+2r-i+1)r!}{(r+1-j)!} \binom{k+r-i+j-1}{j-1} \right). \end{aligned}$$

Formula (4) for $r+1$ instead of r follows by expanding the binomial coefficients. □

Finally, we shall need the following structure result. For completeness, we provide a short proof that should convince that the theory requires E_2.

Proposition 8. *Quasimodular forms can be expressed as linear combinations of derivatives of modular forms and E_2:*

$$\tilde{M}_k^{\leq k/2} = \bigoplus_{i=0}^{k/2-1} \mathrm{D}^i M_{k-2i} \oplus \mathbb{C}\, \mathrm{D}^{k/2-1} E_2.$$

Proof. We proceed by descent on the depth. If f has weight k and depth s, we would like to have a modular form g such that $f - \mathrm{D}^s g$ has depth strictly less than s. For any $g \in M_{k-2s}$, multiple use of differentiation theorem [5, Lemme 118] lead to

$$Q_s(D^s g) = \left(\frac{1}{2\pi\mathrm{i}} \right)^s s! \binom{k-s-1}{s} g. \tag{6}$$

If $\binom{k-s-1}{s} \neq 0$, which happens if $s < k/2$, we can choose

$$g = (2\pi\mathrm{i})^s \frac{(k-2s-1)!}{(k-s-1)!} Q_s(f) \in M_{k-2s}.$$

For $s = k/2$, we use

$$Q_{k/2}(D^{k/2-1}E_2) = \left(\frac{1}{2\pi\mathrm{i}}\right)^{k/2-1}\left(\frac{k}{2}-1\right)!\frac{6}{\pi\mathrm{i}}$$

and choose

$$\alpha = \frac{\pi\mathrm{i}}{6}\cdot\frac{(2\pi\mathrm{i})^{k/2-1}}{\left(\frac{k}{2}-1\right)!}Q_{k/2}(f) \in M_0 = \mathbb{C}$$

to obtain

$$f - \alpha\,\mathrm{D}^{k/2-1}\,E_2 \in \tilde{M}_k^{\leq k/2-1}.$$

□

1.2 Rankin-Cohen brackets for modular forms

Cohen [4] introduced the Rankin-Cohen brackets after a work of Rankin [8–10]. These are bilinear differential operators, whose main property is to preserve modular forms. More precisely, let Γ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. We write $M_k(\Gamma)$ for the space of modular forms of weight k over Γ. For each $n \geq 0$, $(f,g) \in M_k(\Gamma)\times M_\ell(\Gamma)$, define the n-Rankin-Cohen bracket of f and g by

$$[f,g]_n = \sum_{r=0}^{n}(-1)^r\binom{k+n-1}{n-r}\binom{\ell+n-1}{r}\mathrm{D}^r f\,\mathrm{D}^{n-r} g. \tag{7}$$

Then $[f,g]_n \in M_{k+\ell+2n}(\Gamma)$. Moreover, if Φ is a bilinear differential operator sending $M_k(\Gamma)\times M_\ell(\Gamma)$ to $M_{k+\ell+2n}(\Gamma)$ for all $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ a finite index subgroup, then (up to constant) $\Phi(f,g) = [f,g]_n$. For an overview of Rankin-Cohen brackets including a proof of these results[1], see for instance [16], [15] or [5].

Rankin-Cohen brackets appear to be useful in various mathematical domains as for instance invariant theory ([12] and [2]) or non-commutative geometry [14].

2. Rankin-Cohen brackets

We prove our main result (Theorem 1). For $n \geq 0$ and any sequence $\mathbf{a} = (a_r)_{0\leq r\leq n}$, the bilinear forms we study take the form

$$\Phi_{\mathbf{a}}(f,g) = \sum_{r=0}^{n} a_r\,\mathrm{D}^r f\,\mathrm{D}^{n-r} g.$$

[1]The uniqueness result needs explanations: it is proved by using only algebraic arguments, the demonstration does not depend on the group Γ or on growth conditions. Of course, it is possible that for some fixed group Γ the uniqueness result does not hold (for instance if $M_k(\Gamma) = \{0\}$!).

We first establish a sufficient condition on $\mathbf{a}$ (Lemma 9). For s, t and n non-negative integers, we introduce the set

$$\mathcal{E}(s,t,n) = \{(u,v,\alpha,\beta) \in \mathbb{Z}^4_{\geq 0} \colon u \leq s, v \leq t, \\ \alpha+\beta \leq u+v+n-s-t-1\}.$$

Lemma 9. *Let $k,\ell \in \mathbb{Z}_{>0}$, $s \in \{0,\dots,\lfloor k/2 \rfloor\}$, $t \in \{0,\dots,\lfloor \ell/2 \rfloor\}$ and $n \in \mathbb{Z}_{>0}$. For $\mathbf{a} = (a_r)_{0\leq r\leq n}$ satisfying*

$$\sum_{r=0}^{n} a_r \binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)! = 0$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$, one has

$$\Phi_{\mathbf{a}}(\tilde{M}_k^{\leq s}, \tilde{M}_\ell^{\leq t}) \subset \tilde{M}_{k+\ell+2n}^{\leq s+t}.$$

Proof. Let $f \in \tilde{M}_k^{\leq s}$ and $g \in \tilde{M}_\ell^{\leq t}$. From Lemma 7 we deduce

$$\begin{aligned}(\Phi_{\mathbf{a}}(f,g) \underset{k+\ell+2n}{|} \gamma) &= \sum_{r=0}^{n} a_r (f^{(r)} \underset{k+2r}{|} \gamma)(g^{(n-r)} \underset{\ell+2(n-r)}{|} \gamma) \\ &= \sum_{i=0}^{s+t+n} C(\mathbf{a};i)(f,g)\,\mathrm{X}(\gamma)^i\end{aligned}$$

with

$$\begin{aligned}&C(\mathbf{a};i)(f,g) \\ &\quad= \sum_{\substack{(i_1,i_2)\in\mathbb{Z}^2_{\geq 0} \\ i_1+i_2=i}} \sum_{r=0}^{n} a_r \sum_{j_1=0}^{r} \left(\frac{1}{2\pi \mathrm{i}}\right)^{j_1} j_1! \binom{r}{j_1}\binom{k+r-i_1+j_1-1}{j_1} \\ &\qquad \times \sum_{j_2=0}^{n-r} \left(\frac{1}{2\pi \mathrm{i}}\right)^{j_2} j_2! \binom{n-r}{j_2}\binom{\ell+n-r-i_2+j_2-1}{j_2} \\ &\qquad \times Q_{i_1-j_1}(f)^{(r-j_1)} Q_{i_2-j_2}(g)^{(n-r-j_2)}. \end{aligned} \tag{8}$$

It follows that $\Phi_{\mathbf{a}}(f,g) \in \tilde{M}_{k+\ell+2n}^{\leq s+t}$ if and only if $C(\mathbf{a}; s+t+i) = 0$ for all $i \in \{1,\dots,n\}$. This is easily seen to be equivalent to

$$\sum_u \sum_v \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2_{\geq 0}\\ \alpha+\beta=n+u+v-s-t-i}} \left(\frac{1}{2\pi\mathrm{i}}\right)^{n-\alpha-\beta} \sum_r a_r (r-\alpha)!(n-r-\beta)!$$
$$\times \binom{r}{\alpha}\binom{n-r}{\beta}\binom{k+r-u-1}{r-\alpha}\binom{\ell+n-r-v-1}{n-r-\beta} Q_u(f)^{(\alpha)} Q_v(g)^{(\beta)}$$
$$= 0$$

for all $i \in \{1, \dots, n\}$, the sets of summation being determined by the binomial coefficients. Hence, $\Phi_{\mathbf{a}}(\tilde{M}_k^{\leq s}, \tilde{M}_\ell^{\leq t}) \subset \tilde{M}_{k+\ell+2n}^{\leq s+t}$ is implied by

$$\sum_r a_r \binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)! = 0 \qquad (9)$$

for all $(u, v, \alpha, \beta) \in \mathcal{E}(s, t, n)$. □

Remark 8. The statement of the previous lemma is in fact an equivalence, if we ask $\Phi_{\mathbf{a}}$ to satisfy $\Phi_{\mathbf{a}}(\tilde{M}_k^{\leq s}(\Gamma), \tilde{M}_\ell^{\leq t}(\Gamma)) \subset \tilde{M}_{k+\ell+2n}^{\leq s+t}(\Gamma)$ for each finite index subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$: indeed for $\{a(u, v, \alpha, \beta)\}$ a non identically zero family of complex numbers, if

$$\Psi\colon (f, g) \mapsto \sum_{(u,v,\alpha,\beta)\in\mathcal{E}(s,t,n)} a(u, v, \alpha, \beta) Q_u(f)^{(\alpha)} Q_v(g)^{(\beta)}$$

satisfy $\Psi(\tilde{M}_k^{\leq s}(\Gamma), \tilde{M}_\ell^{\leq t}(\Gamma)) = 0$, then there exists $M > 0$ such that the minimum of $\dim(\tilde{M}_k^{\leq s}(\Gamma))$ and $\dim(\tilde{M}_\ell^{\leq t}(\Gamma))$ is strictly smaller than M. However, as for modular forms, for each $A > 0$, there exists a finite index subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$ such that $\dim \tilde{M}_k^{\leq s}(\Gamma) > A$ and $\dim \tilde{M}_\ell^{\leq t}(\Gamma) > A$ (recall that $k, \ell \in \mathbb{Z}_{>0}$).

We shall now give a necessary condition for $\mathbf{a}$ satisfying the condition of Lemma 9.

Lemma 10. *Let $k, \ell \in \mathbb{Z}_{>0}$, $s \in \{0, \dots, \lfloor k/2 \rfloor\}$, $t \in \{0, \dots, \lfloor \ell/2 \rfloor\}$ and $n \in \mathbb{Z}_{>0}$. If $\mathbf{a} = (a_r)_{0\leq r\leq n}$ satisfies*

$$\sum_{r=0}^{n} a_r \binom{r}{\alpha}\binom{n-r}{\beta}(k+r-u-1)!(\ell+n-r-v-1)! = 0$$

for all $(u, v, \alpha, \beta) \in \mathcal{E}(s, t, n)$, then there exists $\lambda \in \mathbb{C}$ such that

$$a_r = \lambda(-1)^r \binom{k+n-s-1}{n-r}\binom{\ell+n-t-1}{r}$$

for all $r \in \{0, \dots, n\}$.

Proof. Define $\mathbf{b} = (b_r)_{0\leq r\leq n}$ by

$$b_r = a_r(k+r-s-1)!(\ell+n-r-t-1)!$$

for all r. Then

$$\sum_{r=0}^{n} b_r \binom{r}{\alpha}\binom{n-r}{\beta}\binom{k+r-u-1}{s-u}\binom{\ell+n-r-v-1}{t-v} = 0$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$. Choosing $u = s$, $t = v$ and $\beta = 0$ leads to $F^{(\alpha)}(1) = 0$ for all $\alpha \in \{0,\dots,n-1\}$ where F is the generating (polynomial) function of $\mathbf{b}$ defined by

$$F(x) = \sum_{r=0}^{n} b_r x^r.$$

This implies the existence of $\mu \in \mathbb{C}$ such that $F(x) = \mu(x-1)^n$ and thus $b_r = \mu(-1)^r\binom{n}{r}$. The result follows by defining

$$\lambda = \mu\frac{n!}{(k-s+n-1)!(\ell-t+n-1)!}.$$

□

We obtain the existence of the Rankin-Cohen operator for quasimodular forms in showing that the vector $\mathbf{a}$ we found in Lemma 10 is admissible.

Lemma 11. *Let* $k,\ell \in \mathbb{Z}_{>0}$, $s \in \{0,\dots,\lfloor k/2\rfloor\}$, $t \in \{0,\dots,\lfloor \ell/2\rfloor\}$ *and* $n \in \mathbb{Z}_{>0}$. *Let* $\mathbf{a} = (a_r)_{1\leq r\leq n}$ *be defined by*

$$a_r = (-1)^r\binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r}.$$

Then

$$\Phi_{\mathbf{a}}(\tilde{M}_k^{\leq s}, \tilde{M}_\ell^{\leq t}) \subset \tilde{M}_{k+\ell+2n}^{\leq s+t}.$$

Proof. By Lemma 9 it suffices to check that

$$\sum_{\substack{(r_1,r_2)\in\mathbb{Z}_{\geq 0}\times\mathbb{Z}_{\geq 0}\\ r_1+r_2=n}} \frac{(-1)^{r_1}}{r_1!r_2!}\binom{r_1}{\alpha}\binom{r_2}{\beta}\binom{k-u-1+r_1}{s-u}\binom{\ell-v-1+r_2}{t-v}$$
$$= 0 \qquad (10)$$

for all $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$. Fix $(u,v,\alpha,\beta) \in \mathcal{E}(s,t,n)$, then (10) is the coefficient of order n in the product $P_1(X)P_2(X)$ where

$$P_1(X) = \sum_{r_1=0}^{+\infty} \frac{(-1)^{r_1}}{r_1!} \binom{r_1}{\alpha} \binom{k-u-1+r_1}{s-u} X^{r_1}$$

$$P_2(X) = \sum_{r_2=0}^{+\infty} \frac{1}{r_2!} \binom{r_2}{\beta} \binom{\ell-v-1+r_2}{t-v} X^{r_2}.$$

We have

$$P_1(X) = \frac{X^\alpha}{\alpha!} Q_1^{(\alpha)}(X)$$

with

$$Q_1(X) = \sum_{r_1=0}^{+\infty} \frac{(-1)^{r_1}}{r_1!} \binom{k-u-1+r_1}{s-u} X^{r_1}$$

and

$$Q_1(X) = \frac{X^{-k+s+1}}{(s-u)!} R_1^{(s-u)}(X)$$

with

$$\begin{aligned} R_1(X) &= \sum_{r_1=0}^{+\infty} \frac{(-1)^{r_1}}{r_1!} X^{r_1+k-u-1} \\ &= X^{k-u-1} e^{-X}. \end{aligned}$$

We therefore may write $P_1(X) = \Pi_1(X)e^{-X}$ where Π_1 is a polynomial of degree $\alpha + s - u$. Similarly, $P_2(X) = \Pi_2(X)e^{X}$ where Π_2 is a polynomial of degree $\beta + t - v$. It follows that $P_1 P_2$ is a polynomial of degree $\alpha + \beta + s + t - u - v$. Finally, since, by definition, $\alpha + \beta - u - v < n - s - t$ we get (10). □

Remark 9. With the help of the hypergeometric methods [7, Chapter 3], we obtain that

$$\Pi_1(X) = (-1)^\alpha \sum_{r=\alpha}^{s-u+\alpha} \binom{k+\alpha-u-1}{k+r-s-1} \binom{r}{\alpha} \frac{X^r}{r!}$$

and

$$\Pi_2(X) = (-1)^\beta \sum_{r=\beta}^{t-v+\beta} (-1)^r \binom{\ell+\beta-v-1}{\ell+r-t-1} \binom{r}{\beta} \frac{X^r}{r!}.$$

Previous lemmas prove Theorem 1.

3. Rankin-Cohen brackets and derivation

In this section, we prove Theorem 3. First, we remark that

$$\begin{aligned}
&\Phi_{n;k,s;\ell,t}(f,g)' \\
&\quad= \sum_{r=0}^{n-1}(-1)^r \left[\binom{k-s+n-1}{n-r}\binom{\ell-t+n-1}{r}\right. \\
&\qquad \left.-\binom{k-s+n-1}{n-r-1}\binom{\ell-t+n-1}{r+1}\right] f^{(r+1)}g^{(n-r)} \\
&\qquad +\binom{k-s+n-1}{n} fg^{(n+1)} + (-1)^n\binom{\ell-t+n-1}{n} f^{(n+1)}g.
\end{aligned} \tag{11}$$

Next,

$$\begin{aligned}
&\Phi_{n;k,s;\ell+2,t+1}(f,g') \\
&\quad= \binom{k-s+n-1}{n} fg^{(n+1)} \\
&\qquad - \sum_{r=0}^{n-1}(-1)^r\binom{k-s+n-1}{n-r-1}\binom{\ell-t+n}{r+1} f^{(r+1)}g^{(n-r)}
\end{aligned}$$

so that

$$\begin{aligned}
&\Phi_{n;k+2,s+1;\ell,t}(f',g) + \Phi_{n;k,s;\ell+2,t+1}(f,g') \\
&\quad= \binom{k-s+n-1}{n} fg^{(n+1)} + (-1)^n\binom{\ell-t+n-1}{n} f^{(n+1)}g \\
&\qquad + \sum_{r=0}^{n-1}(-1)^r \left[\binom{k-s+n}{n-r}\binom{\ell-t+n-1}{r}\right. \\
&\qquad \left.-\binom{k-s+n-1}{n-r-1}\binom{\ell-t+n}{r+1}\right] f^{(r+1)}g^{(n-r)}
\end{aligned} \tag{12}$$

and equality from (11) and (12) follows by expanding the binomial coefficients.

4. A more precise structure result

In this section, we prove Proposition 2. Let $n > 0$. If $f \in \tilde{M}_k^s$ and $g \in \tilde{M}_\ell^t$ then $\Phi_{n;k,s;\ell,t}(f,g)$ has weight $k+\ell+2n$ and depth less than $s+t$. Since

$n > 0$ this depth is not maximal since

$$s + t \leq \frac{k}{2} + \frac{\ell}{2} < \frac{k + \ell + 2n}{2}.$$

Then it follows from Proposition 8 that

$$\Phi_{n;k,s;\ell,t}(f, g) \in M_{k+\ell+2n} \oplus \bigoplus_{j=1}^{s+t} \mathrm{D}^j\, M_{k+\ell+2n-2j}.$$

However, the definition of $\Phi_{n;k,s;\ell,t}(f, g)$ implies that its Fourier coefficient at 0 is 0 and since this is also true for derivatives of modular forms we get

$$\Phi_{n;k,s;\ell,t}(f, g) \in S_{k+\ell+2n} \oplus \bigoplus_{j=1}^{s+t} \mathrm{D}^j\, M_{k+\ell+2n-2j}.$$

The contribution to $\Phi_{n;k,s;\ell,t}(f, g)$ coming from

$$S_{k+\ell+2n} \oplus \bigoplus_{j=1}^{s+t-1} \mathrm{D}^j\, M_{k+\ell+2n-2j}$$

has depth strictly less than $s + t$. Hence

$$Q_{s+t}(\Phi_{n;k,s;\ell,t}(f, g)) = Q_{s+t}(\mathrm{D}^{s+t}\, g)$$

where $g \in M_{k+\ell+2n-2s-2t}$. Since

$$Q_{s+t}(\mathrm{D}^{s+t}\, g) = (2\pi\mathrm{i})^{-s-t} \frac{(k + \ell + 2n - s - t - 1)!}{(k + \ell + 2n - 2s - 2t - 1)!} g$$

(see (6)), to prove that g is parabolic we shall prove that the Fourier coefficient at 0 of $Q_{s+t}(\Phi_{n;k,s;\ell,t}(f, g))$ is 0. From (8) we get

$$\begin{aligned} &Q_{s+t}(\Phi_{n;k,s;\ell,t}(f, g)) \\ &\quad = \sum_u \sum_v \sum_{\substack{(\alpha,\beta)\in\mathbb{Z}^2_{\geq 0} \\ \alpha+\beta=n+u+v-s-t}} \left(\frac{1}{2\pi\mathrm{i}}\right)^{n-\alpha-\beta} \sum_r a_r (r-\alpha)! \\ &\qquad \times (n-r-\beta)! \binom{r}{\alpha} \binom{n-r}{\beta} \binom{k+r-u-1}{r-\alpha} \binom{\ell+n-r-v-1}{n-r-\beta} \\ &\qquad \times Q_u(f)^{(\alpha)} Q_v(g)^{(\beta)}. \end{aligned} \tag{13}$$

Since derivatives of quasimodular forms have Fourier coefficients vanishing at 0, the only contribution to the Fourier coefficient of

$$Q_{s+t}(\Phi_{n;k,s;\ell,t}(f, g))$$

at 0 is given by $(\alpha,\beta)=(0,0)$ in (13). However, the summation is on (α,β) such that $\alpha+\beta=n+u+v-s-t$ and we have $n+u+v-s-t>0$ if $n>s+t$.

Thanks to (13) we also see that if $f\in\tilde{M}_k^{\leq s}$ and $g\in\tilde{M}_\ell^{\leq t}$ satisfies $s+t>0$ and $\hat{g}(0)=0$ then

$$\Phi_{s+t;k,s;\ell,t}(f,g)\in S_{k+\ell+2s+2t}\oplus\bigoplus_{j=1}^{s+t-1}\mathrm{D}^j M_{k+\ell+2s+2t-2j}\oplus\mathrm{D}^{s+t}S_{k+\ell}.$$

5. Applications

An easy but useful consequence of the fact that $\mathrm{D}\,\Delta=\Delta E_2$ is the following lemma.

Lemma 12. *Let $n\geq 0$. Let $f\in\tilde{M}_k^{\leq s}$ and $g\in\tilde{M}_\ell^{\leq t}$. There exists $h\in\tilde{M}_{k+\ell+2n}^{\leq s+t}$ such that*

$$\Phi_{n;k,s;\ell,t}(f,\Delta g)=\Delta h.$$

For example, we have

$$\Phi_{1;k+12,s;12,0}(\Delta f,\Delta)=\Delta\Phi_{1;k,s;12,0}(f,\Delta).$$

5.1 Homogeneous products of derivatives of E_2

In this section we prove Proposition 4 by recursion on n. For $n=0$ we have $E_2^2=E_4+12\,\mathrm{D}\,E_2\in\mathbb{C}E_4\oplus\mathbb{C}\,\mathrm{D}\,E_2$. Assume that:

$$\mathrm{D}^r E_2\,\mathrm{D}^{n-r}E_2\in\bigoplus_{\substack{j=0\\ j\equiv n\pmod 2}}^{n-4}\mathrm{D}^j S_{2n+4-2j}\oplus\mathbb{C}\,\mathrm{D}^n E_4\oplus\mathbb{C}\,\mathrm{D}^{n+1}E_2$$
$$(0\leq r\leq n).$$

Deal first with the case where $n=2m$ is even. By recursion hypothesis, we have

$$\begin{aligned}\mathrm{D}(\mathrm{D}^r E_2\,\mathrm{D}^{n-r}E_2)&=\mathrm{D}^r E_2\,\mathrm{D}^{n+1-r}E_2+\mathrm{D}^{r+1}E_2\,\mathrm{D}^{n-r}E_2\\&\in\bigoplus_{\substack{j=0\\ j\equiv n\pmod 2}}^{n-4}\mathrm{D}^{j+1}S_{2n+4-2j}\oplus\mathbb{C}\,\mathrm{D}^{n+1}E_4\oplus\mathbb{C}\,\mathrm{D}^{n+2}E_2.\end{aligned}$$

The set $\{\mathrm{D}^r E_2 \mathrm{D}^{n-r} E_2,\ 0 \le r \le n\}$ has $m+1$ distinct terms (corresponding to $0 \le r \le m$). The set $\{\mathrm{D}^r E_2 \mathrm{D}^{n+1-r} E_2,\ 0 \le r \le n+1\}$ has also $m+1$ distinct terms (corresponding to $0 \le r \le m$). It follows that

$$\{\mathrm{D}^r E_2 \mathrm{D}^{n+1-r} E_2 + \mathrm{D}^{r+1} E_2 \mathrm{D}^{n-r} E_2,\ r \in \{0,\dots,m\}\}$$

and

$$\{\mathrm{D}^r E_2 \mathrm{D}^{n+1-r} E_2,\ r \in \{0,\dots,m\}\}$$

are basis of the same space with change of basis matrix given by

$$\begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix}.$$

It follows that for any $r \in \{0,\dots,m\}$ (hence any $r \in \{0,\dots,n\}$) we have

$$\mathrm{D}^r E_2 \mathrm{D}^{n+1-r} E_2 \in \bigoplus_{\substack{j=0 \\ j \equiv n+1 \pmod 2}}^{n-3} \mathrm{D}^j S_{2n+6-2j} \oplus \mathbb{C}\,\mathrm{D}^{n+1} E_4 \oplus \mathbb{C}\,\mathrm{D}^{n+2} E_2.$$

We now deal with the case where $n = 2m-1$ is odd. Again, by recursion hypothesis, we have

$$\begin{aligned} \mathrm{D}(\mathrm{D}^r E_2 \mathrm{D}^{n-r} E_2) &= \mathrm{D}^r E_2 \mathrm{D}^{n+1-r} E_2 + \mathrm{D}^{r+1} E_2 \mathrm{D}^{n-r} E_2 \\ &\in \bigoplus_{\substack{j=0 \\ j \equiv n \pmod 2}}^{n-4} \mathrm{D}^{j+1} S_{2n+4-2j} \oplus \mathbb{C}\,\mathrm{D}^{n+1} E_4 \oplus \mathbb{C}\,\mathrm{D}^{n+2} E_2. \end{aligned}$$

The subspace generated by all the quasimodular forms $\mathrm{D}^r E_2 \mathrm{D}^{n+1-r} E_2 + \mathrm{D}^{r+1} E_2 \mathrm{D}^{n-r} E_2$ when r runs over $\{0,\dots,2m-1\}$ is the hyperplane

$$\left\{ \sum_{r=0}^{2m} \alpha_r \mathrm{D}^r E_2 \mathrm{D}^{2m-r} E_2 | \sum_{r=0}^{2m} (-1)^r \alpha_r = 0 \right\}$$

hence it is sufficient for the proof of our recursion step to find a linear combination

$$\sum_{r=0}^{2m} \alpha_r \mathrm{D}^r E_2 \mathrm{D}^{2m-r} E_2 \in \bigoplus_{\substack{j=0 \\ j \text{ even}}}^{2m-4} \mathrm{D}^j S_{4m+4-2j} \oplus \mathbb{C}\,\mathrm{D}^{2m} E_4 \oplus \mathbb{C}\,\mathrm{D}^{2m+1} E_2$$

with

$$\sum_{r=0}^{2m}(-1)^r\alpha_r \neq 0.$$

This is the step where we use Rankin-Cohen brackets. Since $[E_2, E_2]_{2m+2} \in \tilde{M}_{4m+8}^{\leq 2}$ we have $Q_2([E_2, E_2]_{2m+2}) \in S_{4m+4}$ (see (13) for the cuspidality). Equation (8) combined with the fact that $Q_1(E_2)$ is constant implies that

$$\begin{aligned} &Q_2([E_2, E_2]_{2m+2}) \\ &\quad = \frac{24}{(2\pi \mathrm{i})^2}(2m+2)\,\mathrm{D}^{2m+1}E_2 + \frac{4}{(2\pi \mathrm{i})^2} \\ &\quad\quad \times \Bigg[\sum_{r=2}^{2m+2}(-1)^r\binom{2m+2}{r}^2\binom{r}{2}\binom{r+1}{2}\mathrm{D}^{r-2}E_2\,\mathrm{D}^{2m+2-r}E_2 \\ &\quad\quad + \sum_{r=1}^{2m+1}(-1)^r\binom{2m+2}{r}^2\binom{r+1}{2}\binom{2m+3-r}{2} \\ &\quad\quad \times \mathrm{D}^{r-1}E_2\,\mathrm{D}^{2m+1-r}E_2\Bigg]. \end{aligned} \tag{14}$$

Let

$$\alpha_r(N) = 2(-1)^r\binom{r}{2}\binom{N}{r}\binom{N}{r-1}(N+1-2r).$$

Remark that $\alpha_r(N)$ is defined for any $r \in \mathbb{Z}$, and vanishes for any $r \notin [2, N]$. Equation (14) gives

$$\begin{aligned} &\sum_{r=2}^{2m+2}\alpha_r(2m+2)\,\mathrm{D}^{r-2}E_2\,\mathrm{D}^{2m+2-r}E_2 \\ &\quad = (2\pi \mathrm{i})^2 Q_2([E_2, E_2]_{2m+2}) - 24(2m+2)\,\mathrm{D}^{2m+1}E_2 \\ &\quad \in S_{4m+4} \oplus \mathbb{C}\,\mathrm{D}^{2m+1}E_2. \end{aligned}$$

Let $\beta_r(N) = (-1)^r\alpha_r(N)$. We prove that

$$A(N) = \sum_{r=2}^{N}(-1)^r\alpha_r(N) = \sum_{r\in\mathbb{Z}}\beta_r(N)$$

is strictly negative (hence differs from 0). Zeilberger's algorithm (*e.g.*, on the open-source computer algebra system Maxima) [7, Chapter 6] provides a function $K(N,r)$ such that[2]

$$2(N+1)(2N-1)\beta_r(N) - N(N-1)\beta_r(N+1) = K(N,r+1)\beta_{r+1}(N) - K(N,r)\beta_r(N).$$

More precisely

$$K(N,r) = \frac{\mathcal{N}(N,r)}{(N-2r+1)(N-r+1)(N-r+2)(N-1)}. \tag{15}$$

where

$$\mathcal{N}(N,r) = (r-2)(r-1)(N+1) \times [3N^3 + 8N^2(1-r) + N(4r^2-6r+3) - 2r^2 + 4r - 2]$$

We deduce the recursive formula

$$\frac{A(N+1)}{A(N)} = \frac{2(N+1)(2N-1)}{N(N-1)}$$

which, since $A(2) = -4$, implies

$$A(N) = -N(N-1)\binom{2N-2}{N-1} < 0.$$

Finally, we have found a function which belongs to the hyperplane. This completes the proof.

5.2 Niebur formula

From Proposition 4 we obtain

$$\Phi_{4;2,1;2,1}(E_2, E_2) \in S_{12} = \mathbb{C}\Delta.$$

The computation of the first coefficients gives $\Phi_{4;2,1;2,1}(E_2, E_2) = -48\Delta$. This is the differential equation proved by Niebur in [6]:

$$2^3 \cdot 3\Delta = 18(\mathrm{D}^2 E_2)^2 + E_2 \mathrm{D}^4 E_2 - 16 \mathrm{D} E_2 \mathrm{D}^3 E_2$$

and comparing the Fourier expansions gives Niebur formula.

[2]Note that no algorithm is needed to check that $K(N,r)$ as defined in (15) works.

5.3 van der Pol formula

From Proposition 2 we obtain

$$\Phi_{1;4,0;6,1}(E_4, \mathrm{D}\, E_4) \in S_{12}.$$

The computation of the first coefficient gives $\Phi_{1;4,0;6,1}(E_4, \mathrm{D}\, E_4) = 960\Delta$. This is the differential equation proved by van der Pol:

$$4E_4 \,\mathrm{D}^2\, E_4 - 5(\mathrm{D}\, E_4)^2 = 960\Delta.$$

It leads to

$$\begin{aligned}\tau(n) &= n^2\sigma_3(n) + 60 \sum_{a+b=n} (4b-5a)b\sigma_3(a)\sigma_3(b) \\ &= n^2\sigma_3(n) + 60 \sum_{a=1}^{n-1} (9a^2 - 13an + 4n^2)\sigma_3(a)\sigma_3(n-a) \\ &= n^2\sigma_3(n) + 60 \sum_{b=1}^{n-1} (9b^2 - 5bn)\sigma_3(a)\sigma_3(n-a)\end{aligned}$$

and the summation of the two last equalities implies the van der Pol formula in its original form [13, eq. (53)]:

$$\tau(n) = n^2\sigma_3(n) + 60 \sum_{a=1}^{n-1} (2n-3a)(n-3a)\sigma_3(a)\sigma_3(n-a).$$

5.4 Chazy equation

Recall that we proved at the end of the introduction that an equation of the shape

$$\alpha E_2 \,\mathrm{D}^2\, E_2 + \beta(\mathrm{D}\, E_2)^2 = \mathrm{D}^3\, E_2$$

has to exist. Coefficients α and β can be computed by identifications of the first Fourier coefficients. Our aim in this section is to give an interpretation of this equation in terms of Rankin-Cohen brackets. We have

$$\Phi_{1;2,1;12,0}(E_2, \Delta) \in \Delta \tilde{M}_4^{\leq 1} = \mathbb{C}\Delta E_4$$

hence

$$\Phi_{1;2,1;12,0}(E_2, \Delta) = \Delta E_4$$

and

$$\Phi_{1;4,0;12,0}(E_4, \Delta) \in \Delta M_6 = \mathbb{C}\Delta E_6$$

hence

$$\Phi_{1;4,0;12,0}(E_4,\Delta) = 4\Delta E_6$$

so that

$$\Phi_{1;16,0;12,0}(\Phi_{1;2,1;12,0}(E_2,\Delta),\Delta) = \Delta\Phi_{1;4,0;12,0}(E_4,\Delta) = 4\Delta^2 E_6.$$

Next we compute

$$\Phi_{1;30,0;12,0}(\Delta^2 E_6,\Delta) = \Delta^2\Phi_{1;6,0;12,0}(E_6,\Delta) \in \Delta^3 M_8 = \mathbb{C}\Delta^3 E_4^2$$

hence

$$\Phi_{1;30,0;12,0}(\Delta^2 E_6,\Delta) = 6\Delta^3 E_4^2 = 6\Delta\Phi_{1;2,1;12,0}(E_2,\Delta)^2$$

and

$$\begin{aligned}\Phi_{1;30,0;12,0}(\Phi_{1;16,0;12,0}(\Phi_{1;2,1;12,0}(E_2,\Delta),\Delta),\Delta)\\ = 24\Delta\Phi_{1;2,1;12,0}(E_2,\Delta)^2.\end{aligned}$$

This is (2). We deduce the usual form of the Chazy equation in the following way. From

$$K := \Phi_{1;2,1;12,0}(E_2,\Delta) = E_2 \operatorname{D}\Delta - 12\operatorname{D}E_2\Delta = \Delta(E_2^2 - 12\operatorname{D}E_2)$$

we get

$$\begin{aligned}L := \Phi_{1;16,0;12,0}(K,\Delta) &= 16K\Delta - 12\operatorname{D}K\Delta\\ &= 4\Delta^2(E_2^3 - 18E_2\operatorname{D}E_2 + 36\operatorname{D}^2 E_2)\end{aligned}$$

and since

$$\begin{aligned}&\Phi_{1;30,0;12,0}(L,\Delta)\\ &\quad= 30L\operatorname{D}\Delta - 12\operatorname{D}L\Delta\\ &\quad= 24\Delta^3(E_2^4 - 24E_2^2\operatorname{D}E_2 + 72E_2\operatorname{D}^2 E_2 + 36(\operatorname{D}E_2)^2 - 72\operatorname{D}^3 E_2)\end{aligned}$$

the equality $\Phi_{1;30,0;12,0}(L,\Delta) = 24\Delta K^2$ gives the Chazy equation

$$2\operatorname{D}^3 E_2 - 2E_2\operatorname{D}^2 E_2 + 3(\operatorname{D}E_2)^2 = 0.$$

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