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YOUNGJU CHOIE, FRANÇOIS DUMAS, FRANÇOIS MARTIN, AND EMMANUEL ROYER

Abstract. The aim of this very short note is to give details on Oberdieck derivation. This is an unpublished companion to the work *Formal deformations of the algebra of Jacobi forms and Rankin-Cohen brackets* by the same authors.

We build a natural derivation on Jacobi forms that extends Serre derivation. Our construction has been influenced by a construction of some differential operator by Oberdieck in [Obe14] and hence we shall call this derivation the Oberdieck derivation (see also [DLM00, GK09, MTZ08]). References for the Weierstraß $\wp$ and ζ functions are [Lan87, Ch. 18], [Sil94, Ch. 1] and [CS17, Ch. 2].

1. Context and notation

Let $\mathcal{H}$ be the Poincaré upper half-plane, that is the set of complex numbers τ with $\Im\mathfrak{m}\,\tau > 0$. Let k be an integer and m a nonnegative integer. The multiplicative group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{Z}^2$ by right multiplication. The semidirect product of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathbb{Z}^2$ with respect to this action is the Jacobi group: $\mathrm{SL}(2,\mathbb{Z})^J = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$. Let $\mathbf{F}$ be the set of functions from $\mathcal{H}\times\mathbb{C}$ to $\mathbb{C}$. Let k and m be two integers. We have the following actions of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathbb{Z}^2$ on $\mathbf{F}$. Let $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, let $(\lambda,\mu)\in\mathbb{Z}^2$, let $\Phi\in\mathbf{F}$, then

$$\Phi|_{k,m}\begin{pmatrix} a & b\\ c & d\end{pmatrix}(\tau,z) = (c\tau+d)^{-k}\mathrm{e}^{-2\mathrm{i}\pi\frac{mcz^2}{c\tau+d}}\Phi\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)$$

$$\Phi\|_m(\lambda,\mu)(\tau,z) = \mathrm{e}^{2\mathrm{i}\pi m(\lambda^2\tau+2\lambda z)}\Phi(\tau,z+\lambda\tau+\mu)$$

for all $(\tau,z)\in\mathcal{H}\times\mathbb{C}$. These two actions induce an action of $\mathrm{SL}(2,\mathbb{Z})^J$ on $\mathbf{F}$ the following way: if $(\gamma,(\lambda,\mu))\in\mathrm{SL}(2,\mathbb{Z})^J$, if $\Phi\in\mathbf{F}$, then we define

$$\Phi[\![_{k,m}(\gamma,(\lambda,\mu)) = \left(\Phi|_{k,m}\gamma\right)\|_m(\lambda,\mu).$$

Explicitly, if $\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$ and $(\lambda,\mu)\in\mathbb{Z}^2$, then

$$f[\![_{k,m}(\gamma,(\lambda,\mu))(\tau,z) = (c\tau+d)^{-k}\exp\left(2\pi\mathrm{i}m\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z\right)\right)f\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right)$$

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for all $(\tau, z) \in \mathcal{H} \times \mathbb{C}$. A function is invariant by the action of $\mathrm{SL}(2,\mathbb{Z})^J$ if and only if it is invariant by both the action of $\mathrm{SL}(2,\mathbb{Z})$ and the action of $\mathbb{Z}^2$.

A Jacobi form of weight k and index m is a holomorphic function $\Phi\colon \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ that is invariant by the action of the Jacobi group and that has a Fourier expansion of the form

$$\Phi(\tau, z) = \sum_{n=0}^{+\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm}} c(n,r) \mathrm{e}^{2\pi \mathrm{i}(n\tau + rz)}. \tag{1.1}$$

The vector space $\mathcal{J}_{k,m}$ of such functions is finite dimensional. We identify functions on $\mathcal{H} \times \mathbb{C}$ that are not depending on the second variable with functions on $\mathcal{H}$ and define

$$\mathcal{J}_{k,0} = \mathcal{M}_k.$$

The space $\mathcal{M}_k$ is the space of holomorphic modular forms of weight k on $\mathrm{SL}(2,\mathbb{Z})$ and we have

$$\mathcal{M}_* = \bigoplus_{\substack{k \in 2\mathbb{Z}_{\geq 0} \\ k \neq 2}} \mathcal{M}_k.$$

The action $[]_{k,0}$ of $\mathrm{SL}(2,\mathbb{Z})^J$ on $\mathcal{J}_{k,0}$ induces an action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{M}_k$. This action is $|_{k,0}$ and we shall simply write $|_k$.

The bigraded algebra

$$\mathcal{J}_{*,*} = \bigoplus_{k,m} \mathcal{J}_{k,m}$$

is *not* finitely generated and hence we introduce the notion of weak Jacobi form.

A weak Jacobi form of weight k and index m is a function invariant by the action of the Jacobi group but with a Fourier expansion of the form

$$\Phi(\tau, z) = \sum_{n=0}^{+\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4nm + m^2}} c(n,r) \mathrm{e}^{2\mathrm{i}\pi(n\tau + rz)}$$

instead of the one given in (1.1). For any given integer $n \geq 0$, the fact that the sum over r is limited to $r^2 \leq 4nm+m^2$ is a consequence of some periodicity of the coefficients [EZ85, p. 105]. The vector space $\widetilde{\mathcal{J}}_{k,m}$ of such functions is still finite dimensional [EZ85, Theorem 9.2]. As a consequence, we obtain that

$$\widetilde{\mathcal{J}}_{k,0} = \mathcal{M}_k.$$

Let $\mathbf{1}$ be the constant function taking value 1 everywhere (of one or two variables, depending on the context). The subgroup of the modular group $\mathrm{SL}(2,\mathbb{Z})$ of elements γ with $\mathbf{1}|_k \gamma = \mathbf{1}$ is

$$\mathrm{SL}(2,\mathbb{Z})_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

The Eisenstein series of weight $k \in \mathbb{Z}_{\geq 4}$ is

$$\mathrm{E}_k(\tau) = \sum_{\gamma \in \mathrm{SL}(2,\mathbb{Z})_\infty \backslash \mathrm{SL}(2,\mathbb{Z})} \mathbf{1}|_k \gamma(\tau) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d)=1}} (c\tau + d)^{-k}.$$

Its Fourier expansion is given in terms of the divisor functions

$$\forall u \in \mathbb{C} \quad \forall n \in \mathbb{Z}^*_{\geq 0} \qquad \sigma_u(n) = \sum_{d|n} d^u$$

by

$$\forall \tau \in \mathcal{H} \qquad \mathrm{E}_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) q^n$$

where $q = \exp(2\pi\mathrm{i}\tau)$ and B_k is the Bernoulli number of order k. We use this Fourier expansion to define an Eisenstein series of weight two:

$$\mathrm{E}_2(\tau) = 1 - 24 \sum_{n=1}^{+\infty} \sigma_1(n) q^n.$$

For all even $k \geq 2$, we shall sometimes use another normalisation:

$$G_k = -\frac{(2\pi\mathrm{i})^k}{k!} B_k \mathrm{E}_k.$$

2. Two intermediate functions

For all $\tau \in \mathcal{H}$, let $\Lambda_\tau = \mathbb{Z} \oplus \tau\mathbb{Z}$. The ζ function associated to Λ_τ is defined by

$$\forall z \in \mathbb{C} - \Lambda_\tau \qquad \zeta(\tau, z) = \frac{1}{z} + \sum_{\substack{\omega \in \Lambda_\tau \\ \omega \neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right).$$

Sometimes, we shall use the notation $\zeta(\Lambda_\tau, z)$ instead of $\zeta(\tau, z)$. The function $z \mapsto \zeta(z, \tau)$ is meromorphic over $\mathbb{C}$. Its poles are the points of Λ_τ and they are simple.

We define J_1 by

$$\forall \tau \in \mathcal{H}, \forall z \in \mathbb{C} - \Lambda_\tau \qquad \mathrm{J}_1(\tau, z) = \frac{1}{2\pi\mathrm{i}} \zeta(\tau, z) + \frac{\pi\mathrm{i}}{6} z \mathrm{E}_2(\tau).$$

To describe the transformation relations satisfied by J_1, we define a function $\mathrm{X}(M)$, for any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$ by

$$\begin{array}{rccc} \mathrm{X}(M) & : & \mathcal{H} \times \mathbb{C} & \to & \mathbb{C} \\ & & (\tau, z) & \mapsto & \dfrac{cz}{c\tau + d}. \end{array}$$

It satisfies

$$\forall (M, N) \in \mathrm{SL}(2, \mathbb{Z})^2 \qquad \mathrm{X}(M)\big|_{1,0} N = \mathrm{X}(MN) - \mathrm{X}(N).$$

***Lemma* 1–** *The function J_1 satisfies the following transformation properties:*

$$\forall (\lambda, \mu) \in \mathbb{Z}^2 \qquad \mathrm{J}_1 \|_0 (\lambda, \mu) = \mathrm{J}_1 - \lambda$$

$$\forall M \in \mathrm{SL}(2, \mathbb{Z}) \qquad \mathrm{J}_1 |_{1,0} M = \mathrm{J}_1 + \mathrm{X}(M).$$

The Fourier expansion of J_1 *is*

$$\mathrm{J}_1(\tau, z) = -\frac{1}{2} + \frac{\xi}{\xi - 1} - \sum_{n=1}^{+\infty} \left(\sum_{d|n} \left(\xi^d - \xi^{-d} \right) \right) q^n$$

where $\xi = \exp(2\pi \mathrm{i} z)$, *valid if* $\xi \neq 1$ *and* $|q| < |\xi| < |q|^{-1}$.

Its Laurent expansion around 0 *is*

$$\mathrm{J}_1(\tau, z) = \frac{1}{2\pi \mathrm{i} z} - \frac{1}{2\pi \mathrm{i}} \sum_{n=0}^{+\infty} \mathrm{G}_{2n+2}(\tau) z^{2n+1}$$

valid for all $\tau \in \mathcal{H}$ *and* z *in any punctured neighborhood of* 0 *containing no point of* Λ_τ.

Proof. We prove the transformation property by the action of $\mathbb{Z}^2$. We have

$$\mathrm{J}_1(\tau, z+\lambda\tau+\mu) - \mathrm{J}_1(\tau, z) = \frac{1}{2\pi \mathrm{i}} \left(\zeta(\tau, z+\lambda\tau+\mu) - \zeta(\tau, z)\right) + \frac{\pi \mathrm{i}}{6}(\lambda\tau+\mu)\mathrm{E}_2(\tau).$$

Let η be the quasi-period map associated to Λ_τ. Then,

$$\zeta(\tau, z+\lambda\tau+\mu) - \zeta(\tau, z) = \eta(\lambda\tau+\mu).$$

The map η is a homomorphism of the group Λ_τ and hence

$$\eta(\lambda\tau+\mu) = \lambda\eta(\tau) + \mu\eta(1).$$

The Legendre relation implies that $\tau\eta(1) - \eta(\tau) = 2\pi\mathrm{i}$ so that

$$\eta(\lambda\tau+\mu) = (\lambda\tau+\mu)\eta(1) - 2\pi\mathrm{i}\lambda.$$

We have also

$$\eta(1) = -\frac{(2\pi\mathrm{i})^2}{12}\mathrm{E}_2(\tau).$$

We deduce

$$\frac{1}{2\pi \mathrm{i}} \left(\zeta(\tau, z+\lambda\tau+\mu) - \zeta(\tau, z)\right) = -\frac{\pi \mathrm{i}}{6}(\lambda\tau+\mu)\mathrm{E}_2(\tau) - \lambda$$

and

$$\mathrm{J}_1(\tau, z+\lambda\tau+\mu) - \mathrm{J}_1(\tau, z) = -\lambda.$$

We prove the transformation property by the action of $\mathrm{SL}(2,\mathbb{Z})$. First, note that if $z \notin \Lambda_\tau$, then $\frac{z}{c\tau+d} \notin \Lambda_{M\tau}$. Let us show that it is sufficient to prove the result for $M \in \{S, T\}$. Let M and N be such that

$$\mathrm{J}_1|_{1,0} M = \mathrm{J}_1 + \mathrm{X}(M) \qquad \text{and} \qquad \mathrm{J}_1|_{1,0} N = \mathrm{J}_1 + \mathrm{X}(N).$$

Then,

$$\begin{aligned}\mathrm{J}_1|_{1,0} MN &= \left(\mathrm{J}_1|_{1,0} M\right)|_{1,0} N = (\mathrm{J}_1 + \mathrm{X}(M))|_{1,0} N = \mathrm{J}_1 + \mathrm{X}(N) + \mathrm{X}(MN) - \mathrm{X}(N)\\ &= \mathrm{J}_1 + \mathrm{X}(MN).\end{aligned}$$

The multiplicative group $\mathrm{SL}(2,\mathbb{Z})$ is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We deduce that if $\mathrm{J}_1|_{1,0} S = \mathrm{J}_1 + \mathrm{X}(S)$ and $\mathrm{J}_1|_{1,0} T = \mathrm{J}_1$ then $\mathrm{J}_1|_{1,0} M = \mathrm{J}_1 + \mathrm{X}(M)$ for all $M \in \mathrm{SL}(2,\mathbb{Z})$.

Let us prove that $\mathrm{J}_1|_{1,0}T = \mathrm{J}_1$. We have

$$\begin{aligned}\mathrm{J}_1(\tau+1,z) &= \frac{1}{2\pi\mathrm{i}}\zeta(\Lambda_{\tau+1},z) + \frac{\pi\mathrm{i}}{6}z\,\mathrm{E}_2(\tau+1)\\ &= \frac{1}{2\pi\mathrm{i}}\zeta(\Lambda_{\tau},z) + \frac{\pi\mathrm{i}}{6}z\,\mathrm{E}_2(\tau) = \mathrm{J}_1(\tau,z)\end{aligned}$$

since $\Lambda_{\tau+1} = \Lambda_\tau$ and E_2 is periodic of period 1.

Finally, let us prove $\mathrm{J}_1|_{1,0}S = \mathrm{J}_1 + \mathrm{X}(S)$. We have

$$\mathrm{J}_1\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = \frac{1}{2\pi\mathrm{i}}\zeta\left(-\frac{1}{\tau},\frac{z}{\tau}\right) + \frac{\pi\mathrm{i}}{6}\frac{z}{\tau}\mathrm{E}_2\left(-\frac{1}{\tau}\right).$$

We compute

$$\begin{aligned}\zeta\left(-\frac{1}{\tau},\frac{z}{\tau},\right) &= \zeta\left(\tau^{-1}\Lambda_\tau,\tau^{-1}z\right) \quad \text{since } \Lambda_{-1/\tau} = \tau^{-1}\Lambda_\tau\\ &= \tau\zeta(\Lambda_\tau,z) \quad \text{by homogeneity}\\ &= \tau\zeta(\tau,z)\end{aligned}$$

and recall that

$$\tau^{-2}\mathrm{E}_2\left(-\frac{1}{\tau}\right) = \mathrm{E}_2(\tau) + \frac{6}{\pi\mathrm{i}}\frac{1}{\tau}.$$

Finally,

$$\tau^{-1}\mathrm{J}_1\left(-\frac{1}{\tau},\frac{z}{\tau},\right) = \frac{1}{2\pi\mathrm{i}}\zeta(z,\tau) + \frac{\pi\mathrm{i}}{6}z\,\mathrm{E}_2(\tau) + \frac{z}{\tau}$$

or, equivalently,

$$\mathrm{J}_1|_{1,0}S = \mathrm{J}_1 + \mathrm{X}(S).$$

The Fourier expansion of J_1 is a consequence of the following expansion for ζ:

$$\frac{1}{2\pi\mathrm{i}}\zeta(\tau,z) = \sum_{n\geq 1}\left(\frac{\xi^{-1}}{1-\xi^{-1}q^n} - \frac{\xi}{1-\xi q^n}\right)q^n - \frac{\mathrm{i}\pi}{6}z\,\mathrm{E}_2(\tau) - \frac{1}{2} - \frac{\xi}{1-\xi}.$$

The Laurent expansion of J_1 is a consequence of the following expansion for ζ:

$$\zeta(\tau,z) = \frac{1}{z} - \sum_{k=1}^{+\infty}\mathrm{G}_{2k+2}(\tau)z^{2k+1}.$$

□

We define the J_2 function by

$$\mathrm{J}_2 = \mathrm{D}_z\mathrm{J}_1 - \frac{1}{12}\mathrm{E}_2 + \mathrm{J}_1^2$$

where $\mathrm{D}_z = \frac{\partial}{2\pi\mathrm{i}\partial z}$. Similarly, we set $\mathrm{D}_\tau = \frac{\partial}{2\pi\mathrm{i}\partial\tau}$.s

***Lemma* 2–** *The function J_2 satisfies the following transformations properties:*

$$\forall(\lambda,\mu)\in\mathbb{Z}^2 \qquad \mathrm{J}_2\|_0(\lambda,\mu) = \mathrm{J}_2 - 2\lambda\mathrm{J}_1 + \lambda^2$$

$$\forall M\in\mathrm{SL}(2,\mathbb{Z}) \qquad \mathrm{J}_2|_{2,0}M = \mathrm{J}_2 + 2\mathrm{J}_1\mathrm{X}(M) + \mathrm{X}(M)^2.$$

The Fourier expansion of J_2 *is*

$$\mathrm{J}_2(\tau,z)=\frac{1}{6}-2\sum_{n=1}^{+\infty}\left(\sum_{d\mid n}\frac{n}{d}\left(\xi^d-\xi^{-d}\right)\right)q^n$$

valid if $|q|<|\xi|<|q|^{-1}$.

Its Laurent expansion around 0 *is*

$$\mathrm{J}_2(\tau,z)=-\frac{2}{(2\pi\mathrm{i})^2}\mathrm{G}_2(\tau)-\sum_{n=0}^{+\infty}\frac{1}{n+1}\mathrm{D}_\tau(\mathrm{G}_{2n+2})(\tau)z^{2n+2}$$

valid for all $\tau\in\mathcal{H}$ *and* z *in any punctured neighborhood of* 0 *containing no point of* Λ_τ.

Proof. To prove the transformation properties, we apply D_z to the transformation relations satisfied by J_1 and get

$$\mathrm{D}_z(\mathrm{J}_1)|_{2,0}M=\mathrm{D}_z(\mathrm{J}_1)+\frac{1}{2\pi\mathrm{i}z}\mathrm{X}(M)$$

and

$$\mathrm{D}_z(\mathrm{J}_1)\|_1(\lambda,\mu)=\mathrm{D}_z(\mathrm{J}_1).$$

The relations for J_2 follow from these equalities and the definition.

From the definition of J_2 and the Laurent expansion of J_1, we have

$$(2\pi\mathrm{i})^2\mathrm{J}_2(\tau,z)=-2\,\mathrm{G}_2(\tau)+\sum_{k\geq 0}\left[-(2k+5)\,\mathrm{G}_{2k+4}(\tau)+\sum_{a+b=k}\mathrm{G}_{2a+2}(\tau)\,\mathrm{G}_{2b+2}(\tau)\right]z^{2k+2}.$$

The Laurent expansion of J_2 follows then from an equality due to Ramanujan (see [Sko93, Eq. (1)]).

As a corollary of the Laurent expansions of J_1 and J_2, we have that $\mathrm{D}_z(\mathrm{J}_2)=2\,\mathrm{D}_\tau(\mathrm{J}_1)$. We get from the Fourier expansion of J_1 the following

$$\mathrm{D}_z(\mathrm{J}_2)(\tau,z)=-2\sum_{n\geq 1}n\sum_{d\mid n}\left(\xi^d-\xi^{-d}\right)q^n=-2\,\mathrm{D}_z\left(\sum_{n\geq 1}\sum_{d\mid n}\frac{n}{d}\left(\xi^d+\xi^{-d}\right)q^n\right).$$

We deduce that a function H exists such that

$$\mathrm{J}_2(\tau,z)=-2\sum_{n\geq 1}\sum_{d\mid n}\frac{n}{d}\left(\xi^d+\xi^{-d}\right)q^n+H(\tau).$$

We have

$$\mathrm{J}_2(\tau,0)=H(\tau)-4\sum_{n\geq 1}\sum_{d\mid n}\frac{n}{d}q^n=H(\tau)+\frac{1}{6}(\mathrm{E}_2(\tau)-1)$$

and hence

$$\mathrm{J}_2(\tau,0)=H(\tau)-\frac{1}{6}-\frac{2}{(2\pi\mathrm{i})^2}\mathrm{G}_2(\tau).$$

The Laurent expansion of J_2 implies

$$\mathrm{J}_2(\tau,0)=-\frac{2}{(2\pi\mathrm{i})^2}\mathrm{G}_2(\tau).$$

We deduce $H(\tau)=1/6$. □

3. Oberdieck's derivation

Let $(k,p)\in 2\mathbb{Z}\times\mathbb{Z}_{\geq 0}$. For $f\in\widetilde{\mathcal{J}}_{k,p}$, let

$$\mathrm{Ob}(f)=\mathrm{D}_\tau(f)-\frac{k}{12}f\,\mathrm{E}_2-\mathrm{J}_1\,\mathrm{D}_z(f)+p\,\mathrm{J}_2\,f.$$

***Proposition* 3–** *For $(k,p)\in 2\mathbb{Z}\times\mathbb{Z}_{\geq 0}$, the map* Ob *is linear from $\widetilde{\mathcal{J}}_{k,p}$ to $\widetilde{\mathcal{J}}_{k+2,p}$. Moreover, if $(\ell,q)\in 2\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ and $(f,g)\in\widetilde{\mathcal{J}}_{k,p}\times\widetilde{\mathcal{J}}_{\ell,q}$ then*

$$\mathrm{Ob}(fg)=\mathrm{Ob}(f)g+f\,\mathrm{Ob}(g).$$

Proof. The computation of $\mathrm{Ob}(fg)$ is left to the reader. Let $f\in\widetilde{\mathcal{J}}_{k,p}$ and $M\in\mathrm{SL}(2,\mathbb{Z})$. We have

$$\mathrm{D}_\tau\left(f|_{k,p}M\right)=\left(p\,\mathrm{X}(M)^2-\frac{k}{2\pi\mathrm{i}z}\mathrm{X}(M)\right)f|_{k,p}M-\mathrm{X}(M)\left(\mathrm{D}_z(f)|_{k+1,p}M\right)+\mathrm{D}_\tau(f)|_{k+2,p}M$$

and

$$\mathrm{D}_z\left(f|_{k,p}M\right)=-2p\,\mathrm{X}(M)\left(f|_{k,p}M\right)+\mathrm{D}_z(f)|_{k+1,p}M.$$

Since $f|_{k,p}M=f$ we deduce

$$\mathrm{D}_\tau(f)|_{k+2,p}M=\mathrm{D}_\tau(f)+\left(\frac{k}{2\pi\mathrm{i}z}\mathrm{X}(M)-p\,\mathrm{X}(M)^2\right)f+\mathrm{X}(M)\left(\mathrm{D}_z(f)|_{k+1,p}M\right)$$

and

$$\mathrm{D}_z(f)|_{k+1,p}M=\mathrm{D}_z(f)+2p\,\mathrm{X}(M)f.$$

In particular,

$$\mathrm{D}_\tau(f)|_{k+2,p}M=\mathrm{D}_\tau(f)+\left(\mathrm{D}_z(f)+\frac{k}{2\pi\mathrm{i}z}f\right)\mathrm{X}(M)+pf\,\mathrm{X}(M)^2. \tag{3.1}$$

From,

$$(\mathrm{J}_1\,\mathrm{D}_z(f))|_{k+2,p}M=\left(\mathrm{J}_1|_{1,0}M\right)\left(\mathrm{D}_z(f)|_{k+1,p}M\right)$$

we get

$$(\mathrm{J}_1\,\mathrm{D}_z(f))|_{k+2,p}M=\mathrm{J}_1\,\mathrm{D}_z(f)+(\mathrm{D}_z(f)+2p\,\mathrm{J}_1\,f)\mathrm{X}(M)+2pf\,\mathrm{X}(M)^2. \tag{3.2}$$

Similarly,

$$\left(-\frac{k}{12}\mathrm{E}_2\,f\right)|_{k+2,p}M=-\frac{k}{12}\mathrm{E}_2\,f-\frac{k}{2\pi\mathrm{i}z}f\,\mathrm{X}(M) \tag{3.3}$$

and

$$(p\,\mathrm{J}_2\,f)|_{k+2,p}M=p\,\mathrm{J}_2\,f+2p\,\mathrm{J}_1\,f\,\mathrm{X}(M)+pf\,\mathrm{X}(M)^2. \tag{3.4}$$

Equations (3.1), (3.2), (3.3) and (3.4) lead to

$$\mathrm{Ob}(f)|_{k+2,p}M=f.$$

Let $(\lambda,\mu)\in\mathbb{Z}^2$. Then

$$\mathrm{D}_z(f)\|_p(\lambda,\mu)=\mathrm{D}_z(f)-2pf\lambda$$

and

$$\mathrm{D}_\tau(f)\|_p(\lambda,\mu)=\mathrm{D}_\tau(f)-\mathrm{D}_z(f)\lambda+pf\lambda^2 \tag{3.5}$$

and so

$$(-\mathrm{J}_1\,\mathrm{D}_z(f))\|_p(\lambda,\mu) = -\mathrm{J}_1\,\mathrm{D}_z(f) + (\mathrm{D}_z(f) + 2pf\,\mathrm{J}_1)\,\lambda - 2pf\,\lambda^2.$$

We also have

$$(p\,\mathrm{J}_2\,f)\|_p(\lambda,\mu) = p\,\mathrm{J}_2\,f - 2p\,\mathrm{J}_1\,f\,\lambda + pf\,\lambda^2. \tag{3.6}$$

Equations (3.5)–(3.6) lead to

$$\mathrm{Ob}(f)\|_p(\lambda,\mu) = f.$$

Finally, let $\tau \in \mathcal{H}$. We prove that $\mathrm{Ob}_\tau\colon z \mapsto \mathrm{Ob}(f)(\tau,z)$ is holomorphic. By invariance by the action of $\mathbb{Z}^2$, it is sufficient to prove that Ob_τ has no pole in $\mathcal{F}_\tau = \{a+b\tau\colon (a,b)\in [0,1[^2\}$. The invariance of f by the action of $\mathrm{SL}(2,\mathbb{Z})$ implies that the Laurent expansion of f around 0 is

$$f(\tau,z) = \sum_{\nu=0}^{+\infty} Q_{2\nu}(\tau) z^{2\nu}$$

where $Q_{2\nu}$ is a quasimodular form of weight $k+2\nu$ and depth less that or equal to ν (see [Roy12], [MR05] or [Zag08]). The lack of odd powers in z is a consequence of the non existence of odd weight quasimodular form. The only pole of ζ in $\mathcal{F}_\tau$ is 0 and so J_1 has no other pole than 0 in $\mathcal{F}_\tau$. The Laurent expansion of J_1 implies that the Laurent expansion of $\mathrm{J}_1\,\mathrm{D}_z f$ around $z=0$ is bounded and hence $\mathrm{J}_1\,\mathrm{D}_z f$ has no pole in $\mathcal{F}_\tau$. The function J_2 has no other pole in $\mathcal{F}_\tau$ than 0 as it can be seen from its definition. The Laurent expansion of J_2 implies than 0 is not a pole. Finally, Ob_τ is holomorphic. □

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