Formal deformations: from modular to Jacobi through quasimodular forms

Hong Kong, July 2018

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Star product (an overview)

Formal deformations: definition

Let A be a commutative algebra. The commutative algebra of formal power series is $A[[\hbar]]$.

Definition

Let $\mu = (\mu_n)_{n\in\mathbb{N}}$ be a sequence of bilinear applications $A \times A \longrightarrow A$ such that μ_0 is the product on A. Define a product on $A[[\hbar]]$ by extension of

$$f \star g = \sum_{n\in\mathbb{N}} \mu_n(f,g)\hbar^n \qquad (f,g \in A).$$

The sequence μ is a formal deformation of A if this non commutative product is associative.

Poisson brackets: an observation

Let $(\mu_n)_{n\in\mathbb{N}}$ be a formal deformation of a commutative algebra A. If μ_1 is skew-symmetric and μ_2 is symmetric, then μ_1 is a Poisson bracket on A.

Definition

Let A be a commutative algebra and $\mu\colon A\times A \longrightarrow A$ a skew-symmetric bilinear application, then (A,μ) is a Poisson algebra if for all a, b and c in A,

$$\mu(ab,c) = a\mu(b,c) + b\mu(a,c) \qquad (\textit{Leibniz})$$

$$\mu\left(a,\mu(b,c)\right) + \mu\left(b,\mu(c,a)\right) + \mu\left(c,\mu(a,b)\right) = 0 \quad (\textit{Jacobi}).$$

Poisson brackets on polynomial algebras

Let A be the commutative algebra $\mathbb{C}[x_1, \dots, x_n]$. Consider a Poisson bracket $[\ ,\ ]$ on A. It is entirely determined by the values $[x_i, x_j]$ for $1 \le i < j \le n$:

$$[P, Q] = \sum_{1 \le i < j \le n} \left(\frac{\partial P}{\partial x_i} \frac{\partial Q}{\partial x_j} - \frac{\partial Q}{\partial x_i} \frac{\partial P}{\partial x_j} \right) [x_i, x_j]$$

Poisson brackets on polynomial algebras

- $A = \mathbb{C}[x]$: the unique Poisson bracket on A is the zero one.
- $A = \mathbb{C}[x, y]$: for any $P \in A$ there exists a Poisson bracket on A defined by $[x, y] = P$.
- $A = \mathbb{C}[x, y, z]$: for any P, Q and R in A, there exists a Poisson bracket on A defined by

$$[x, y] = R, \quad [y, z] = P, \quad [z, x] = Q$$

if and only if

$$(P, Q, R) \cdot \mathrm{curl}(P, Q, R) = 0$$

where

$$\mathrm{curl}(P, Q, R) = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Bidifferential Poisson bracket : solvable case

Let d and δ be two derivations of a commutative algebra A. We assume that $d \circ \delta - \delta \circ d = 2\delta$.

Theorem (Connes & Moscovici, 2004)

The sequence $(\mu_n)_{n\in\mathbb{N}}$ of maps $\mu_n : A \times A \longrightarrow A$ defined by

$$\mu_n(f,g) = \sum_{r=0}^{n} \frac{(-1)^r}{r!(n-r)!} \delta^r \left((d + r\,\mathrm{id})^{\langle n-r \rangle} (f) \right) \cdot \delta^{n-r} \left((d + (n-r)\,\mathrm{id})^{\langle r \rangle} (g) \right)$$

is a formal deformation of A.

Notation: $\phi^{\langle n \rangle} = \phi \circ (\phi + \mathrm{id}) \circ (\phi + 2\,\mathrm{id}) \circ \cdots \circ (\phi + (n-1)\,\mathrm{id})$
Corollary: $[f,g] = \mu_1(f,g) = d(f)\delta(g) - \delta(f)d(g)$ defines a Poisson bracket on A.

Formal deformation of graded algebras

Let $A = \bigoplus_{k \geq 0} A_k$ be a graded commutative algebra. Let d be any derivation of A satisfying $d(A_k) \subset A_{k+2}$.

Theorem (Zagier, 1994)

The sequence $(\mu_n)_{n \in \mathbb{N}}$ of maps $\mu_n : A \times A \to A$ defined by

$$\mu_n(f, g) = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r} \binom{\ell+n-1}{r} d^r(f) d^{n-r}(g)$$

($f \in A_k, g \in A_\ell$) is a formal deformation of A.

(Quasi)modular forms (an overview)

Modular forms: definition

A modular form of weight k is a holomorphic function on $\mathcal{H} = \{z \in \mathbb{C} \colon Imz > 0\}$ satisfying

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z) \quad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z})$$

and having a Fourier expansion

$$f(z) = \sum_{n=0}^{+\infty} \widehat{f}(n) e^{2\pi i n z}.$$

The set of modular forms of weight k is a finite dimensional vector space. If is non zero if and only if $k \in 2\mathbb{N}$, $k \neq 2$.

Modular forms: Eisenstein series

For any $k \in 2\mathbb{N}$, let

$$\mathrm{E}_k(z) = -\frac{k!}{(2i\pi)^k B_k} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k}.$$

This series is absolutely convergent if $k \geq 4$ and admits a Fourier expansion:

$$\mathrm{E}_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{+\infty} \sigma_{k-1}(n) e^{2i\pi nz}$$

where

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

It defines a modular form of weight k.

Modular forms: structure

The set of modular forms of weight k is a finite dimensional vector space $\mathcal{M}_k$. For any k and ℓ, we have $\mathcal{M}_k\mathcal{M}_\ell \subset \mathcal{M}_{k+\ell}$ and, $\mathcal{M}_k \cap \mathcal{M}_\ell = \{0\}$ if $k \neq \ell$. The weight defines a graduation over the algebra $\mathcal{M}$ of all modular forms.

Theorem

The Eisenstein series E_4 *and* E_6 *are algebraically independent and generate* $\mathcal{M}$*.*

$$\mathcal{M} = \mathbb{C}[E_4, \mathrm{E}_6] = \bigoplus_{k\in 2\mathbb{N},\, k\neq 2} \mathcal{M}_k \text{ with } \mathcal{M}_k = \bigoplus_{4i+6j=k} \mathbb{C}\,\mathrm{E}_4^i\,\mathrm{E}_6^j\,.$$

Modular forms: derivatives

Let $\mathrm{D}(f) = \frac{1}{2\pi i} f'(z)$. Let f be a modular form of weight k. Then

$$(cz+d)^{-(k+2m)}\,\mathrm{D}^m(f)\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{m} \binom{m}{j} \frac{(k+m-1)!}{(k+m-j-1)!} \left(\frac{1}{2i\pi}\right)^j \mathrm{D}^{m-j}(f)(z) \left(\frac{c}{cz+d}\right)^j . \quad (1)$$

The derivative of a modular form is not a modular form.

Quasimodular forms: definition

Definition

A quasimodular form of weight k and depth s is a holomorphic function f on $\mathcal{H}$ such that there exist holomorphic functions $f_0, \ldots, f_s$ with $f_s \neq 0$ satisfying

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{n=0}^{s} f_n(z) \left(\frac{c}{cz+d}\right)^n$$

and such that each f_n has a Fourier expansion with only positive indices.

We define $\widetilde{\mathcal{M}}$ the algebra of all quasimodular forms. The function f_0 is necessarily f and hence a quasimodular form of depth 0 is a modular form. If f is a quasimodular form of weight k and depth s, then $\mathrm{D}(f)$ is a quasimodular form of weight $k+2$ and depth $s+1$.

Quasimodular forms: from Eisenstein series

The algebra $\mathcal{M} = \mathbb{C}[E_4, E_6]$ is not stable by complex derivation since

$$D(E_4) = \frac{1}{3}(E_4 E_2 - E_6)$$

$$D(E_6) = \frac{1}{2}\left(E_6 E_2 - E_4^2\right).$$

However,

$$D(E_2) = \frac{1}{12}\left(E_2^2 - E_4\right)$$

hence the algebra $\mathbb{C}[E_2, E_4, E_6]$ is stable by derivation.

Quasimodular forms: structure

$$\widetilde{\mathcal{M}} = \mathbb{C}[\mathrm{E}_2, \mathrm{E}_4, \mathrm{E}_6]$$

$$\widetilde{\mathcal{M}} = \bigoplus_{k \in 2\mathbb{N}} \mathcal{M}_k^{\leq k/2}, \qquad \mathcal{M}_k^{\leq s} = \bigoplus_{j=0}^{s} \mathcal{M}_{k-2j}\, \mathrm{E}_2^j\,.$$

An element in $\mathcal{M}_k^{\leq s}$ is a polynomial in E_2 with coefficients in $\mathcal{M}$. The degree in E_2 is the depth of this element, k is its weight.

$$\widetilde{\mathcal{M}} = \bigoplus_{k \in 2\mathbb{N}} \bigoplus_{s=0}^{k/2} \mathcal{M}_k^s, \qquad \mathcal{M}_k^s = \mathcal{M}_{k-2s}\, \mathrm{E}_2^s = \bigoplus_{\substack{(i,j) \in \mathbb{N}^2 \\ 4i+6j+2s=k}} \mathbb{C}\, \mathrm{E}_4^i\, \mathrm{E}_6^j\, \mathrm{E}_2^s$$

Rankin-Cohen brackets: modular forms

☞The derivative of a modular form is in general not a modular form. Can we find combinations of derivatives preserving the modularity?

In the seventies, Henri Cohen constructed bi-differential operators that send modular forms to modular forms. Let $n \in \mathbb{N}$, the n-th Rankin-Cohen bracket is defined by bilinear extension of the following definition on homogeneous components: if $f \in \mathcal{M}_k$ and $g \in \mathcal{M}_\ell$ then

$$\mathrm{RC}_n(f,g) = \sum_{j=0}^{n} (-1)^j \binom{k+n-1}{n-j} \binom{\ell+n-1}{j} \mathrm{D}^j(f)\, \mathrm{D}^{n-j}(g).$$

We have

$$\mathrm{RC}_n\left(\mathcal{M}_k, \mathcal{M}_\ell\right) \subset \mathcal{M}_{k+\ell+2n}.$$

Rankin-Cohen brackets: modular forms

In the nineties, Don Zagier begun to study the algebraic properties of Rankin-Cohen brackets. With Paula Cohen and Yuri Manin, using a combinatorial equality proved later by Yi-Jun Yao, he proved that the sequence $(\mathrm{RC}_n)_{n\in\mathbb{N}}$ is a formal deformation of the algebra $\mathcal{M}$.

In particular, $\mathcal{M}$ is a Poisson algebra for the bracket $[\ ,\ ] = \mathrm{RC}_1$ defined by $[\mathrm{E}_4, \mathrm{E}_6] = -2(\mathrm{E}_4^3 - \mathrm{E}_6^2) = -2\Delta$.

Rankin-Cohen brackets on quasimodular forms?

For $\mathbf{a}(n,k,\ell,s,t) = \left(a_j(n,k,\ell,s,t)\right)_{j\in\mathbb{N}}$, let us define

$$\mathrm{RC}_n^{\mathbf{a},k,\ell,s,t}(f,g) = \sum_{j=0}^{n} a_j(n,k,\ell,s,t)\,\mathrm{D}^j(f)\,\mathrm{D}^{n-j}(g)$$

for $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$. In general, we have

$$\mathrm{RC}_n^{\mathbf{a},k,\ell,s,t}\left(\mathcal{M}_k^s, \mathcal{M}_\ell^t\right) \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t+n}$$

In order to mimic Rankin-Cohen brackets on modular forms, can we find $\mathbf{a}(n,k,\ell,s,t)$ such that:

$$\mathrm{RC}_n^{\mathbf{a},k,\ell,s,t}\left(\mathcal{M}_k^s, \mathcal{M}_\ell^t\right) \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}?$$

Rankin-Cohen brackets on quasimodular forms?

With François Martin, I proved that the only (up to multiplicative constant) possibility is to define

$$\mathrm{RC}_n(f,g) = \sum_{j=0}^{n} (-1)^j \binom{k-s+n-1}{n-j} \binom{\ell-t+n-1}{j} \mathrm{D}^j(f)\,\mathrm{D}^{n-j}(g).$$

if $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$. Does this define a formal deformation on $\widetilde{\mathcal{M}}$?

Quasimodular algebra is not Poisson for RC_1!

Let us compute

$$p = RC_1(E_4, E_6) = -2(E_4^3 - E_6^2)$$
$$q = RC_1(E_6, E_2) = \frac{1}{2} E_2 E_4^2 - \frac{1}{2} E_4 E_6$$
$$r = RC_1(E_2, E_4) = -\frac{1}{3} E_2 E_6 + \frac{1}{3} E_4^2 .$$

We obtain

$$\mathrm{curl}(p, q, r).(p, q, r) = \frac{1}{12} E_4 p \neq 0.$$

It follows that RC_1 cannot be extended to give $\widetilde{\mathcal{M}}$ a structure of Poisson algebra and that $(RC_n)_{n\geq 0}$ does not define a formal deformation of $\widetilde{\mathcal{M}}$.

Main problem and results for quasimodular forms

Strategy

To construct formal deformations of $\widetilde{\mathcal{M}}$, I used with François Dumas the following strategy.

- Construction of 'all' the Poisson structures on $\widetilde{\mathcal{M}}$ that extend the one given to $\mathcal{M}$ by RC_1.
- Classification of these structures up to Poisson isomorphism.
- Construction of differential expressions of these structures to build some formal deformations.

Isomorphisms

Definition

Let b be a Poisson bracket on $\widetilde{\mathcal{M}}$. An algebra isomorphism φ on $\widetilde{\mathcal{M}}$ is a Poisson modular isomorphism if

$$\varphi\left(\mathrm{b}(f, g)\right) = \mathrm{b}\left(\varphi(f), \varphi(g)\right)$$

and

$$\varphi(\mathcal{M}) \subset \mathcal{M}.$$

Indeed we have the following Poisson rigidity result: the restriction to $\mathcal{M}$ of a Poisson modular isomorphism is the identity.

Notation: $A \overset{\text{🐟}}{\simeq} B$.

First family

Proposition

For any $\lambda \in \mathbb{C}^*$*, there exists an admissible Poisson bracket* $\{\ ,\ \}_\lambda$ *on* $\widetilde{\mathcal{M}}$ *defined by:* $\{E_4, E_6\}_\lambda = -2\Delta$ *and*

$$\{E_2, E_4\}_\lambda = -\frac{1}{3}\left(2\,E_6\,E_2 - \lambda\,E_4^2\right)$$

$$\{E_2, E_6\}_\lambda = -\frac{1}{2}\left(2\,E_4^2\,E_2 - \lambda\,E_4\,E_6\right).$$

Moreover,

$$\forall(\lambda, \lambda') \in \mathbb{C}^{*2} \qquad \left(\widetilde{\mathcal{M}}, \{\ ,\ \}_\lambda\right) \simeq \left(\widetilde{\mathcal{M}}, \{\ ,\ \}_{\lambda'}\right).$$

Second family

Proposition

For any $\alpha \in \mathbb{C}$, *there exists an admissible Poisson bracket* $(\ ,\ )_\alpha$ *on* $\widetilde{\mathcal{M}}$ *defined by:* $(E_4, E_6)_\alpha = -2\Delta$ *and*

$$(E_2, E_4)_\alpha = \alpha\, E_6\, E_2$$

$$(E_2, E_6)_\alpha = \frac{3}{2}\alpha\, E_4^2\, E_2\,.$$

Moreover,

$$\left(\widetilde{\mathcal{M}}, (\ ,\ )_\alpha\right) \simeq \left(\widetilde{\mathcal{M}}, (\ ,\ )_{\alpha'}\right) \Leftrightarrow \alpha = \alpha'.$$

Third family

Proposition

For any $\mu \in \mathbb{C}$, there exists an admissible Poisson bracket $\langle\ ,\ \rangle_\mu$ on $\widetilde{\mathcal{M}}$ defined by $\langle \mathrm{E}_4, \mathrm{E}_6\rangle_\mu = -2\Delta$ and:

$$\begin{aligned} \langle \mathrm{E}_2, \mathrm{E}_4\rangle_\mu &= 4\,\mathrm{E}_6\,\mathrm{E}_2 + \mu\,\mathrm{E}_4^2 \\ \langle \mathrm{E}_2, \mathrm{E}_6\rangle_\mu &= 6\,\mathrm{E}_4^2\,\mathrm{E}_2 - 2\mu\,\mathrm{E}_4\,\mathrm{E}_6\,. \end{aligned}$$

Moreover,

$$\forall (\mu, \mu') \in \mathbb{C}^{*2} \qquad \left(\widetilde{\mathcal{M}}, \langle\ ,\ \rangle_\mu\right) \simeq \left(\widetilde{\mathcal{M}}, \langle\ ,\ \rangle_{\mu'}\right).$$

Classification

Our classification is complete in the following sense.

Theorem

Up to Poisson modular isomorphism, the only distinct admissible Poisson brackets on $\widetilde{\mathcal{M}}$ *are* $\{\ ,\ \}_1$, $\langle\ ,\ \rangle_1$ *and the family* $(\ ,\ )_\alpha$ *for any* $\alpha \in \mathbb{C}$.

☞ If $\alpha = 4$ we have $(\ ,\ )_4 = \langle\ ,\ \rangle_0$.

Differential expressions

A strategy to define formal deformations is to find derivations ∂ on $\widetilde{\mathcal{M}}$ and maps $\kappa\colon \mathbb{N}^2 \to \mathbb{C}$ such that the Poisson brackets we constructed have the form:

$$\mathrm{b}(f,g) = \kappa(k,s) f\,\partial(g) - \kappa(\ell,t) g\,\partial(f) \quad (f \in \mathcal{M}_k^s,\ g \in \mathcal{M}_\ell^t)$$

and to define

$$\mathrm{b}_n(f,g) = \sum_{j=0}^{n} (-1)^j \binom{\kappa(k,s)+n-1}{n-j}\binom{\kappa(\ell,t)+n-1}{j} \partial^j(f)\,\partial^{n-j}(g).$$

We shall restrict to derivations ∂ that act on depth and weight like the complex derivation: $\partial \mathcal{M}_k^{\leq s} \subset \mathcal{M}_{k+2}^{\leq s+1}$.

Differential expressions: first family

Let us define a derivation w on $\widetilde{\mathcal{M}}$ by setting

$$\mathrm{w}(f) = \frac{\{\Delta, f\}_1}{12\Delta}.$$

For $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$ we have

$$\{f, g\}_1 = kf\,\mathrm{w}(g) - \ell g\,\mathrm{w}(f).$$

The set of derivations ∂ on $\widetilde{\mathcal{M}}$ such that $\partial\mathcal{M}_k^{\leq s} \subset \mathcal{M}_{k+2}^{\leq s+1}$ and $kf\,\partial(g) - \ell g \partial(f) = 0$ for all $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$, for all k, ℓ, s, t is a one-dimensional vector space generated by π defined by

$$\pi(f) = kf\,\mathrm{E}_2 \quad (f \in \mathcal{M}_k^{\leq\infty}).$$

For $a \in \mathbb{C}$, let $\mathrm{d}_a = a\pi + \mathrm{w}$.

Formal deformation from the first family

Theorem

For any $a \in \mathbb{C}$, the brackets defined for any integer $n \geq 0$ by

$$[f,g]_{\mathrm{d}_a,n} = \sum_{r=0}^{n} (-1)^r \binom{k+n-1}{n-r}\binom{\ell+n-1}{r} \mathrm{d}_a^r(f)\, \mathrm{d}_a^{n-r}(g)$$

for $f \in \mathcal{M}_k^{\leq\infty}$ and $g \in \mathcal{M}_\ell^{\leq\infty}$ satisfy

$$\left[\mathcal{M}_k^{\leq\infty}, \mathcal{M}_\ell^{\leq\infty}\right]_{\mathrm{d}_a,n} \subset \mathcal{M}_{k+\ell+2n}^{\leq\infty}$$

and define a formal deformation of $\widetilde{\mathcal{M}}$.
Moreover, $\left[\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}\right]_{\mathrm{d}_a,n} \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}$ for all n, s, t, k, ℓ if and only if $a = 0$.

Differential expressions: second family

Let us define a derivation w_α on $\widetilde{\mathcal{M}}$ by setting

$$\mathrm{w}_\alpha(f) = \frac{(\Delta, f)_\alpha}{12\Delta}.$$

For $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$ we have

$$(f, g)_\alpha = [k - (3\alpha + 2)s]f\,\mathrm{w}_\alpha(g) - [\ell - (3\alpha + 2)t]g\,\mathrm{w}_\alpha(f).$$

The set of derivations ∂ on $\widetilde{\mathcal{M}}$ such that $\partial\mathcal{M}_k^{\le s} \subset \mathcal{M}_{k+2}^{\le s+1}$ and $[k - (3\alpha + 2)s]f\,\partial(g) - [\ell - (3\alpha + 2)t]g\,\partial(f) = 0$ for all $f \in \mathcal{M}_k^s$ and $g \in \mathcal{M}_\ell^t$, for all k, ℓ, s, t is a one-dimensional vector space generated by π_α defined by

$$\pi_\alpha(f) = [k - (3\alpha + 2)s]f\,\mathrm{E}_2 \quad (f \in \mathcal{M}_k^s).$$

For $b \in \mathbb{C}$, let $\mathrm{d}_{\alpha,b} = b\pi_\alpha + \mathrm{w}_\alpha$.

Formal deformation from the second family

Theorem

The brackets defined for any integer $n \geq 0$ by

$$[f, g]_{\mathrm{d}_{\alpha,b},n} = \sum_{r=0}^{n} (-1)^r \binom{k - (3\alpha + 2)s + n - 1}{n - r} \binom{\ell - (3\alpha + 2)t + n - 1}{r} \times \mathrm{d}_{\alpha,b}^{r}(f)\, \mathrm{d}_{\alpha,b}^{n-r}(g),$$

for any $f \in \mathcal{M}_k^s, g \in \mathcal{M}_\ell^t$ define a formal deformation of $\widetilde{\mathcal{M}}$ satisfying $\left[\mathcal{M}_k^{\leq s}, \mathcal{M}_\ell^{\leq t}\right]_{\mathrm{d}_{\alpha,b},n}^{\mathcal{K}} \subset \mathcal{M}_{k+\ell+2n}^{\leq s+t}$ if and only if $b = 0$.

And for (weak) Jacobi forms?

Weak Jacobi forms

Let k be an even integer and m be a non negative integer. A weak Jacobi form is a holomorphic function $\Phi\colon \mathcal{H} \to \mathbb{C}$ such that

- If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$ then

$$\Phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{2i\pi \frac{mcz^2}{c\tau+d}} \Phi(\tau, z)$$

- if $(\lambda, \mu) \in \mathbb{Z}^2$, then

$$\Phi(\tau, z + \lambda\tau + \mu) = e^{-2i\pi m(\lambda^2\tau + 2\lambda z)} \Phi(\tau, z)$$

- Φ has a Fourier expansion

$$\Phi(\tau, z) = \sum_{n=0}^{+\infty} \sum_{r \in \mathbb{Z}} c(n, r) e^{2i\pi(n\tau + rz)}.$$

Bigraded structure

The vector space $\widetilde{\mathcal{J}}_{k,m}$ is a finite dimensional space. We consider the algebra of weak Jacobi forms:

$$\widetilde{\mathcal{J}} = \bigoplus_{\substack{k \in 2\mathbb{Z} \\ m \in \mathbb{N}}} \widetilde{\mathcal{J}}_{k,m}.$$

This is a polynomial algebra that we describe.

Eisenstein series

The Eisenstein series of weight $k \geq 4$ and index m is

$$\mathrm{E}_{k,m}(\tau, z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \sum_{\lambda\in\mathbb{Z}} (c\tau + d)^{-k} e^{2i\pi m\left(\lambda^2 \frac{a\tau+b}{c\tau+d} + \frac{2\lambda z - cz^2}{c\tau+d}\right)}.$$

We have

$$\mathrm{E}_{k,m} \in \widetilde{\mathcal{J}}_{k,m}.$$

Generators

Let

$$\Phi_{10,1} = \frac{1}{144}(E_6\, E_{4,1} - E_4\, E_{6,1}),\ \Phi_{12,1} = \frac{1}{144}(E_4^2\, E_{4,1} - E_6\, E_{6,1}).$$

Let

$$A = \frac{\Phi_{10,1}}{\Delta} \in \widetilde{\mathcal{J}}_{-2,1} \quad \text{and} \quad B = \frac{\Phi_{12,1}}{\Delta} \in \widetilde{\mathcal{J}}_{0,1}.$$

Then

$$\widetilde{\mathcal{J}} = \mathbb{C}[E_4, E_6, A, B] = \mathcal{M}[A, B].$$

Localization

We localize the algebra $\widetilde{\mathcal{J}}$ with respect to A:

$$\widetilde{\mathcal{K}} = \mathbb{C}[E_4, E_6, A, A^{-1}, B]$$

and set

$$F_2 = \frac{B}{A}$$

to get

$$\widetilde{\mathcal{K}} = \mathbb{C}[E_4, E_6, F_2][A, A^{-1}].$$

The algebra

$$\widetilde{\mathcal{Q}} = \mathbb{C}[E_4, E_6, F_2]$$

is isomorphic to $\widetilde{\mathcal{M}}$.

Different algebras

$$
\begin{array}{ccc}
\widetilde{\mathcal{J}} = \mathbb{C}[E_4, E_6, A, B] & \hookrightarrow & \widetilde{\mathcal{K}} = \mathbb{C}[E_4, E_6, A^{\pm 1}, B] \\
\uparrow & \nearrow & \uparrow \\
\mathcal{M} = \mathbb{C}[E_4, E_6] & \hookrightarrow & \widetilde{\mathcal{Q}} = \mathbb{C}[E_4, E_6, F_2] \simeq \widetilde{\mathcal{M}}
\end{array}
$$

Serre Rankin-Cohen brackets

Serre's derivation is a derivation on $\mathcal{M}$ defined by

$$\forall f \in \mathcal{M}_k \qquad \mathrm{Se}(f) = f - \frac{k}{12} f\,\mathrm{E}_2\,.$$

We use it to build Serre-Rankin-Cohen brackets:

$$\mathrm{SRC}_n(f,g) = \sum_{j=0}^{n} (-1)^j \binom{k+n-1}{n-j} \binom{\ell+n-1}{j} \mathrm{Se}^j(f)\,\mathrm{Se}^{n-j}(g).$$

We get a formal deformation of $\mathcal{M}$.

Extended Serre Rankin-Cohen brackets

$$
\begin{array}{ccc}
\widetilde{\mathcal{J}} = \mathbb{C}[E_4, E_6, A, B] & \hookrightarrow & \widetilde{\mathcal{K}} = \mathbb{C}[E_4, E_6, A^{\pm 1}, B] \\
\uparrow & \nearrow & \uparrow \\
\mathcal{M} = \mathbb{C}[E_4, E_6] & \hookrightarrow & \widetilde{\mathcal{Q}} = \mathbb{C}[E_4, E_6, F_2] \simeq \widetilde{\mathcal{M}}
\end{array}
$$

We define a formal deformation of $\widetilde{\mathcal{J}}$ that extends $(\mathrm{SRC}_n)_{n\in\mathbb{N}}$.

Extended Serre Rankin-Cohen brackets

We generalize Serre's derivation:

$$\mathrm{Se}_{a,b}(\mathrm{E}_4) = -\frac{1}{3}\,\mathrm{E}_6 \qquad \mathrm{Se}_{a,b}(\mathrm{E}_6) = -\frac{1}{2}\,\mathrm{E}_4^2,$$

$$\mathrm{Se}_{a,b}(\mathrm{A}) = a\,\mathrm{B} \qquad \mathrm{Se}_{a,b}(\mathrm{B}) = b\,\mathrm{E}_A\,\mathrm{A}\,.$$

Define

$$\{f,g\}_n^{[a,b,c]} = \sum_{r=0}^{n} (-1)^r \binom{k+cp+n-1}{n-r} \binom{\ell+cq+n-1}{r} \mathrm{Se}_{a,b}^r(f)\,\mathrm{Se}_{a,b}^{n-r}(g)$$

for all homogeneous $f \in \widetilde{\mathcal{J}}_{k,p}$ and $g \in \widetilde{\mathcal{J}}_{\ell,q}$.
The restriction of $\{\cdot,\cdot\}_n^{[a,b,c]}$ to $\mathcal{M}$ is SRC_n.

Extended Serre Rankin-Cohen brackets

Theorem

For all $(a, b, c) \in \mathbb{C}^3$*, the sequence* $\left(\{\cdot,\cdot\}_n^{[a,b,c]}\right)_{n\in\mathbb{N}}$ *is a formal deformation of* $\widetilde{\mathcal{J}}$ *that satisfies*

$$\{\widetilde{\mathcal{J}}_{k,p}, \widetilde{\mathcal{J}}_{\ell,q}\}_n^{[a,b,c]} \subset \widetilde{\mathcal{J}}_{k+\ell+2,p+q}$$

for all $(k, p, \ell, q, n) \in 2\mathbb{Z} \times \mathbb{N} \times 2\mathbb{Z} \times \mathbb{N} \times \mathbb{N}$*.*

Localized version of Rankin-Cohen brackets

$$
\begin{array}{ccc}
\widetilde{\mathcal{J}} = \mathbb{C}[E_4, E_6, A, B] & \hookrightarrow & \widetilde{\mathcal{K}} = \mathbb{C}[E_4, E_6, A^{\pm 1}, B] \\
\uparrow & \nearrow & \uparrow \\
\mathcal{M} = \mathbb{C}[E_4, E_6] & \hookrightarrow & \widetilde{\mathcal{Q}} = \mathbb{C}[E_4, E_6, F_2] \simeq \widetilde{\mathcal{M}}
\end{array}
$$

We extend the formal deformation on $\widetilde{\mathcal{J}}$ to $\widetilde{\mathcal{K}}$ in two ways and recover, by restriction, the formal deformations of $\widetilde{\mathcal{M}}$.

Localized version of Rankin-Cohen brackets: 1

Let d_α the derivation on $\widetilde{\mathcal{K}}$ defined by

$$d_\alpha(\mathrm{E}_4) = -\frac{1}{3}\,\mathrm{E}_6 + 4\alpha\,\mathrm{E}_4\,\mathrm{F}_2 \qquad d_\alpha(\mathrm{E}_6) = -\frac{1}{2}\,\mathrm{E}_4^2 + 6\alpha\,\mathrm{E}_6\,\mathrm{F}_2,$$

$$d_\alpha(\mathrm{A}) = -2\alpha\,\mathrm{A}\,\mathrm{F}_2 \qquad d_\alpha(\mathrm{F}_2) = -\frac{1}{12}\,\mathrm{E}_4 + 2\alpha\,\mathrm{F}_2^2\,.$$

Let $\left([\cdot,\cdot]_n^{\alpha,c}\right)_{n\in\mathbb{N}}$ be defined by

$$[f,g]_n^{\alpha,c} = \sum_{i=0}^{n}(-1)^i\binom{k+cp+n-1}{n-i}\binom{\ell+cq+n-1}{i}d_\alpha^i(f)d_\alpha^{n-i}(g),$$

for all homogeneous $f \in \widetilde{\mathcal{K}}_{k,p}$ and $g \in \widetilde{\mathcal{K}}_{\ell,q}$.

Localized version of Rankin-Cohen brackets: 1

Proposition

1. *The sequence* $\left([\cdot,\cdot]_n^{\alpha,c}\right)_{n\in\mathbb{N}}$ *is a formal deformation of* $\widetilde{\mathcal{K}}$*,*
2. $[\widetilde{\mathcal{K}}_{k,p},\widetilde{\mathcal{K}}_{\ell,q}]_n^{\alpha,c}\subset\widetilde{\mathcal{K}}_{k+\ell+2n,p+q}$,
3. *the subalgebra* $\widetilde{\mathcal{Q}}$ *is stable by* $\left([\cdot,\cdot]_n^{\alpha,c}\right)_{n\in\mathbb{N}}$*, and the formal deformation* $\left(\widetilde{\mathcal{Q}},\left([\cdot,\cdot]_n^{\alpha,c}\right)_n\right)$ *is isomorphic to the formal deformation* $\left(\widetilde{\mathcal{M}},\left([\cdot,\cdot]_{d_\alpha,n}\right)_n\right)$*,*
4. *if* $\alpha=0$*, the subalgebra* $\widetilde{\mathcal{J}}$ *is stable by* $\left([\cdot,\cdot]_n^{0,c}\right)_{n\in\mathbb{N}}$*, and the restriction of* $\left([\cdot,\cdot]_n^{0,c}\right)_{n\in\mathbb{N}}$ *to* $\widetilde{\mathcal{J}}$ *is the deformation* $\left(\{\cdot,\cdot\}_n^{[0,b,c]}\right)_{n\in\mathbb{N}}$ *of* $\widetilde{\mathcal{J}}$ *for* $b=-\frac{1}{12}$*(and then up to equivalence for any* $b\in\mathbb{C}^\times$*).*

Localized version of Rankin-Cohen brackets: 2

Let δ_β the derivation on $\widetilde{\mathcal{K}}$ defined by

$$\delta_\beta(\mathrm{E}_4) = -\frac{1}{3}\,\mathrm{E}_6 + 4\beta\,\mathrm{E}_4\,\mathrm{F}_2 \qquad \delta_\beta(\mathrm{E}_6) = -\frac{1}{2}\,\mathrm{E}_4^2 + 6\beta\,\mathrm{E}_6\,\mathrm{F}_2,$$
$$\delta_\beta(\mathrm{A}) = -2\beta\,\mathrm{A}\,\mathrm{F}_2 \qquad \delta_\beta(\mathrm{F}_2) = 2\beta\,\mathrm{F}_2^2\,.$$

Let $\left(\langle\cdot,\cdot\rangle_n^{\beta,c}\right)_{n\in\mathbb{N}}$ be defined by

$$\langle f,g\rangle_n^{\beta,c} = \sum_{i=0}^{n}(-1)^i\binom{k+cp+n-1}{n-i}\binom{\ell+cq+n-1}{i}\delta_\beta^i(f)\delta_\beta^{n-i}(g),$$

for all homogeneous $f \in \widetilde{\mathcal{K}}_{k,p}$ and $g \in \widetilde{\mathcal{K}}_{\ell,q}$.

Localized version of Rankin-Cohen brackets: 2

Proposition

(i) *The sequence* $\left(\langle\cdot,\cdot\rangle_n^{\beta,c}\right)_{n\in\mathbb{N}}$ *is a formal deformation of* $\widetilde{\mathcal{K}}$,

(ii) $\langle\widetilde{\mathcal{K}}_{k,p},\widetilde{\mathcal{K}}_{\ell,q}\rangle_n^{\beta,c}\subset\widetilde{\mathcal{K}}_{k+\ell+2n,p+q}$,

(iii) *the subalgebra* $\widetilde{\mathcal{Q}}$ *is stable by* $\left(\langle\cdot,\cdot\rangle_n^{\beta,c}\right)_{n\in\mathbb{N}}$, *and the formal deformation* $\left(\widetilde{\mathcal{Q}},\left(\langle\cdot,\cdot\rangle_n^{\beta,c}\right)_n\right)$ *is isomorphic to the formal deformation* $\left(\widetilde{\mathcal{M}},\left([\cdot,\cdot]_{\delta_{-2/3,0},n}^{\mathcal{K}_{-2/3}}\right)_n\right)$,

(iv) *if* $\beta=0$, *the subalgebra* $\widetilde{\mathcal{J}}$ *is stable by* $\left(\langle\cdot,\cdot\rangle_n^{0,c}\right)_{n\in\mathbb{N}}$, *and the restriction of* $\left(\langle\cdot,\cdot\rangle_n^{0,c}\right)_{n\in\mathbb{N}}$ *to* $\widetilde{\mathcal{J}}$ *is the deformation* $\left(\{\cdot,\cdot\}_n^{[0,0,c]}\right)_{n\in\mathbb{N}}$ *of* $\widetilde{\mathcal{J}}$.

From modular to localized version

$$
\begin{array}{ccc}
\widetilde{\mathcal{J}} = \mathbb{C}[E_4, E_6, A, B] & \hookrightarrow & \widetilde{\mathcal{K}} = \mathbb{C}[E_4, E_6, A^{\pm 1}, B] \\
\uparrow & \nearrow & \uparrow \\
\mathcal{M} = \mathbb{C}[E_4, E_6] & \hookrightarrow & \widetilde{\mathcal{Q}} = \mathbb{C}[E_4, E_6, F_2] \simeq \widetilde{\mathcal{M}}
\end{array}
$$

We extend the formal deformation on $\mathcal{M}$ given by the Rankin-Cohen brackets to $\widetilde{\mathcal{K}}$.

From modular to localized version

Let ∂_u be the derivation of $\widetilde{\mathcal{K}}$ defined by

$$\partial_u(\mathrm{E}_4) = -\frac{1}{3}(\mathrm{E}_6 - \mathrm{E}_4\,\mathrm{F}_2) \qquad \partial_u(\mathrm{E}_6) = \frac{1}{2}(\mathrm{E}_4^2 - \mathrm{E}_6\,\mathrm{F}_2)$$

$$\partial(\mathrm{F}_2) = -\frac{1}{12}(\mathrm{E}_4 - \mathrm{F}_2^2) \qquad \partial_u(\mathrm{A}) = u\,\mathrm{A}\,\mathrm{F}_2\,.$$

For any complex parameters u and v, let $\left([\![\cdot,\cdot]\!]_n^{u,v}\right)_{n\in\mathbb{N}}$ be defined by

$$[\![f,g]\!]_n^{u,v} = \sum_{r=0}^{n}(-1)^r\binom{k+vp+n-1}{n-r}\binom{\ell+vq+n-1}{r}\partial_u^r(f)\,\partial_u^{n-r}(g),$$

for all homogeneous $f \in \widetilde{\mathcal{K}}_{k,p}$ and $g \in \widetilde{\mathcal{K}}_{\ell,q}$.

From modular to localized version...

Theorem

For all $(u, v) \in \mathbb{C}^2$,

1. *the sequence* $\left(\llbracket\cdot,\cdot\rrbracket_n^{u,v}\right)_{n\in\mathbb{N}}$ *is a formal deformation of* $\widetilde{\mathcal{K}}$,
2. $\llbracket\widetilde{\mathcal{K}}_{k+,p}, \widetilde{\mathcal{K}}_{\ell,q}\rrbracket_n^{u,v} \subset \widetilde{\mathcal{K}}_{k+\ell+2n,p+q}$,
3. *the sequence* $\left(\llbracket\cdot,\cdot\rrbracket_n^{u,v}\right)_{n\in\mathbb{N}}$ *restricts to the formal deformation of the algebra* $\mathcal{M}$ *of modular forms given by the usual Rankin-Cohen brackets.*

… and back to Jacobi forms

Lemma

The algebra $\widetilde{\mathcal{J}}$ is stable by the Poisson bracket $[\![\cdot,\cdot]\!]_1^{u,v}$ if and only if $v - 1 = 12u$.

Conjecture

For any complex number u, the sequence $\left([\![\cdot,\cdot]\!]_n^{u,12u+1}\right)_{n\in\mathbb{N}}$ is a formal deformation of the algebra $\widetilde{\mathcal{J}}$ of weak Jacobi forms.

Thanks

THANK YOU!